Invariant measures for the dyadic transformation

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Definition 1.

Let $(X, \mathcal{A})$ be a measurable space and $S\colon X \to X$ be a measurable transformation. A measure μ on $(X, \mathcal{A})$ is said to be invariant for S if

$$\mu(S^{-1}(A)) = \mu(A) \quad \text{for every } A \in \mathcal{A}.$$

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Find all Borel probability measures that are invariant for the dyadic transformation.

Lemma 1.

1. If μ is an invariant measure for the dyadic transformation, then the formula

$$\varphi(x) = \mu([0, x]) \tag{1}$$

   defines an increasing and right-continuous function $\varphi\colon [0, 1] \to [0, 1]$ such that $\varphi(1) = 1$ and

$$\varphi(x) = \varphi\left(\frac{x}{2}\right) + \varphi\left(\frac{x+1}{2}\right) - \varphi\left(\frac{1}{2}\right) \quad \text{for every } x \in [0, 1]. \tag{2}$$

2. If $\varphi\colon [0, 1] \to [0, 1]$ is an increasing and right-continuous function satisfying (2) with $\varphi(1) = 1$, then formula (1) determines uniquely an invariant measure for the dyadic transformation.

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2. If $\varphi\colon [0, 1] \to [0, 1]$ is an increasing and right-continuous function satisfying (2) with $\varphi(1) = 1$, then formula (1) determines uniquely an invariant measure for the dyadic transformation.

Proof.

$$\varphi(x) = \mu\left(\left[0, \tfrac{x}{2}\right] \cup \left(\tfrac{1}{2}, \tfrac{x+1}{2}\right]\right) = \mu\left(\left[0, \tfrac{x}{2}\right]\right) + \mu\left(\left[0, \tfrac{x+1}{2}\right]\right) - \mu\left(\left[0, \tfrac{1}{2}\right]\right)$$
$$= \varphi\left(\tfrac{x}{2}\right) + \varphi\left(\tfrac{x+1}{2}\right) - \varphi\left(\tfrac{1}{2}\right)$$

Describe the set

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and define $T\colon\mathcal{I}\to\mathcal{I}$ by putting

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Given a Banach limit B and $\phi \in \mathcal{I}$ we associate with them $B_\phi \in \mathcal{I}$ defined by

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Theorem 1.

Let B be a Banach limit. Then $\mathcal{S} = \{B_\phi \,|\, \phi \in \mathcal{I}\}$.

Proof.

If $\varphi \in \mathcal{S}$, then

$$B_\varphi(x) = B((T^m\varphi(x))_{m\in\mathbb{N}}) = B((\varphi(x))_{m\in\mathbb{N}}) = \varphi(x)$$

for every $x \in [0,1]$, and hence $\mathcal{S} \subset \{B_\phi \,|\, \phi \in \mathcal{I}\}$.

Fix $\phi \in \mathcal{I}$. Clearly, $B_\phi \in \mathcal{I}$. Moreover, for every $x \in [0,1]$ we have

$$\begin{aligned}
B_\phi(x) &= B\left((T^m\phi(x))_{m\in\mathbb{N}}\right) = B\left(\left(T^{m+1}\phi(x)\right)_{m\in\mathbb{N}}\right)\\
&= B\left(\left(T^m\phi\left(\frac{x}{2}\right) + T^m\phi\left(\frac{x+1}{2}\right) - T^m\phi\left(\frac{1}{2}\right)\right)_{m\in\mathbb{N}}\right)\\
&= B\left(\left(T^m\phi\left(\frac{x}{2}\right)\right)_{m\in\mathbb{N}}\right) + B\left(\left(T^m\phi\left(\frac{x+1}{2}\right)\right)_{m\in\mathbb{N}}\right)\\
&\quad - B\left(\left(T^m\phi\left(\frac{1}{2}\right)\right)_{m\in\mathbb{N}}\right) = B_\phi\left(\frac{x}{2}\right) + B_\phi\left(\frac{x+1}{2}\right) - B_\phi\left(\frac{1}{2}\right).
\end{aligned}$$

Lemma 2.

Every $\varphi \in \mathcal{S}$ is a convex combination of functions of the following three classes:

$$\mathcal{S}_A = \{\varphi \in \mathcal{S} \,|\, \varphi \text{ is an absolutely continuous function}\},$$

$$\mathcal{S}_C = \{\varphi \in \mathcal{S} \,|\, \varphi \text{ is a continuous and singular function}\},$$

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Theorem 2 (A.Rényi, 1957).

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Prove or disprove that if B is a Banach limit and $\phi \in \mathcal{I}$ is continuous and singular, then B_ϕ is continuous and singular.

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Each $\varphi \in \mathcal{S}$ is continuous at all points outside of the set

$$D = \left\{ \frac{p}{2^k - 1} \,\middle|\, p \in \{1, \dots, 2^k - 1\}, k \in \mathbb{N} \right\}.$$

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Decompose D on subsets D_n, $n \in \mathbb{N}$, where each D_n is the smallest subset of D with the following property: If $\varphi \in \mathcal{S}$ is discontinuous at a point of D_n, then it is discontinuous at all points of D_n. We can proceed as follows:
for $k = 1$ we put $D_1 = \{1\}$; for $k = 2$ we put $D_2 = \{\frac{1}{3}, \frac{2}{3}\}$;
for $k = 3$ we put $D_3 = \{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$, $D_4 = \{\frac{3}{7}, \frac{6}{7}, \frac{5}{7}\}$;
for $k = 4$ we put $D_6 = \{\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}\}$, $D_7 = \{\frac{3}{15}, \frac{6}{15}, \frac{12}{15}, \frac{9}{15}\}$, $D_2 = \{\frac{5}{15}, \frac{10}{15}\}$,
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Next for every $n \in \mathbb{N}$ we define a right-continuous jump function $\varphi_n \colon [0,1] \to [0,1]$ such that $\varphi_n(0) = 0$, $\varphi_n(1) = 1$, and φ_n has equal jumps at every point of D_n.

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Theorem 4.

$\mathcal{S}_J=\left\{\sum_{n\in\mathbb{N}}\alpha_n\varphi_n\,\middle|\,\alpha_n\geq 0 \text{ for } n\in\mathbb{N} \text{ with } \sum_{n\in\mathbb{N}}\alpha_n=1\right\}.$

For $m \in \mathbb{N}$ and a probability vector $P = (p_0, \ldots, p_{2^m-1}) \in [0,1)^{2^m}$ let $\Phi_P \colon [0,1] \to [0,1]$ be the unique increasing and continuous function satisfying

$$\Phi_P\left(\frac{x+k}{2^m}\right) = p_k \Phi_P(x) + \sum_{i=0}^{k-1} p_i \quad \text{for } k \in \{0, \ldots, 2^m - 1\}.$$

Then define $\varphi_P \colon [0,1] \to [0,1]$ by

$$\varphi_P(x) = \frac{1}{m} \sum_{i=0}^{m-1} \sum_{k=0}^{2^i-1} \left[\Phi_P\left(\frac{x+k}{2^i}\right) - \Phi_P\left(\frac{k}{2^i}\right)\right]$$

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Theorem 5.

Let Δ_m be the set of all probability vectors $P \in [0,1)^{2^m}$. Then for every Borel probability measure ν on Δ_m the formula

$$\Psi_\nu(x) = \int_{\Delta_m} \varphi_P(x)\, d\nu(P)$$

defines a function $\Psi_\nu \in \mathcal{S}_C$.