Some new results on integrals in the L^r-setting.

Paul Musial

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Joint work with V. Skvortsov, P. Sworowski and F. Tulone

The Promised Land Symposium
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I realize that I am standing between you and supper

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A New Descriptive Characterization of the HK_r-Integral and its Inclusion in Burkill's Integrals, submitted to JMAA, 2024.

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   - Tampering
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5. Coefficients Problem
    - CP-integral
    - $ACG_r = ACG_r^*$
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    - HK_r vs. Wide Denjoy

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AC-function

F is an AC-function on E if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $\sum_{i=1}^{n} |F(I_i)| < \varepsilon$ whenever $\{I_i\}_{i=1}^{n}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n} |I_i| < \eta$.

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It is quite natural to then play with the definition of AC in an attempt to obtain descriptive characterizations of other types of integrals.

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The η condition must not be tampered with.

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Unfortunately for our sophomore, even the function $F(x) = x$ is not an ACG_{wise}-function on $[a, b]$. This is because at least one of the sets E_n has positive outer measure. For any interval $I, F(I) = |I|$ and so regardless of the gauge, by Vitali's lemma, we can find a δ-fine division in $[a, b]$ such that $\sum_{i=1}^n |F(I_i)| > C\mu^*(E_n)$ for some constant C.

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Once again we see that the η condition is essential. Is the δ-fineness also essential? We shall see a bit later that it is when we discuss generalized AC functions.

L^r-Derivative

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Definition: L^r-derivative

A function $F \in L^r[a,b]$ is said to be L^r-differentiable at $x \in (a,b)$ if there exists a real number α such that

$$\left(\frac{1}{h}\int_{-h}^{h} |F(x+t) - F(x) - \alpha t|^r \, dt\right)^{1/r} = o(h)$$

as $h \to 0^+$. The number α is the L^r-derivative of F at x.

L^r-Derivative, alternative definition

Using integration by parts, it can be shown that the L^r-derivative can also be defined as follows:

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$$\sum_{i=1}^{n} \left(\frac{1}{|I_i|} \int_{I_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

We then set

$$(HK_r) \int_a^b f = F(b) - F(a)$$

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The HK_r-integral does indeed integrate all L^r-derivatives and strictly contains the HK-integral.

ACG_r-functions

In the same paper PM and YS gave the following definition:

Definition of ACG_r

F is AC_r on E if for all $\varepsilon > 0$ there exist $\eta > 0$ and a gauge δ defined on $[a, b]$ so that for any δ-fine division $\{(I_i, x_i)\}_{i=1}^q$ in $[a, b]$ having tags in E such that $\sum_{i=1}^q |I_i| < \eta$ we have

$$\sum_{i=1}^{q} \left(\frac{1}{|I_i|} \int_{I_i} |F(y) - F(x_i)|^r dy \right)^{1/r} < \varepsilon.$$

F is an ACG_r-function on $[a, b]$ if $[a, b]$ can be written as countable union of sets on each of which F is an AC_r-function.

A Gap – Bridged

Characterization of HK_r-integral

A function f is HK_r-integrable on $[a, b]$ if and only if there exists $F \in ACG_r[a, b]$ such that $F'_r = f$ a.e.; the function $F(x) - F(a)$ being the indefinite HK_r-integral of f.

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Unfortunately, there was a gap. Whereas AC-functions and ACG_δ-functions possess a derivative a.e., PM and YS did not prove that an ACG_r-function possesses an L^r-derivative a.e. Therefore the above statement provided only a *partial* descriptive characterization of the HK_r-integral.

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This gap was bridged in 2023:

ACG_r-functions possess an L^r-derivative a.e. (PM, Skvortsov, Sworowski, Tulone, submitted to Math. Sbornik, 2024)

If F is an ACG_r-function on $[a,b]$, then F possesses an L^r-derivative a.e. on $[a,b]$.

ACG-functions

Approximate derivative

Let $F : [a,b] \to \mathbb{R}$ and let $c \in (a,b)$. F is approximately differentiable at c if there is a measurable set E such that c is a point of density of E and

$$\lim_{x \to c, x \in E} \frac{F(x) - F(c)}{x - c}$$

exists. This limit is denoted by $F'_{\text{ap}}(x)$.

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ACG-function, $[ACG]$-function

F is an AC-function on E if for each $\varepsilon > 0$ there exists $\eta > 0$ such that $\sum_{i=1}^{n} |F(I_i)| < \varepsilon$, whenever $\{I_i\}_{i=1}^{n}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy $\sum_{i=1}^{n} |I_i| < \eta$. F is an ACG-function on E if E can be written as countable union of sets on each of which F is an AC function.

If, moreover, E can be written as a countable union of closed sets on each of which F is an AC function, then we say that F is an $[ACG]$-function on E.

Wide Denjoy and Kubota Integrals

Wide Denjoy integral

A function $f\colon [a,b] \to \mathbb{R}$ is said to be *D-integrable* (*integrable in the wide Denjoy sense*) if there exists a *continuous* ACG-function $F\colon [a,b] \to \mathbb{R}$ such that $F'_{\mathrm{ap}}(x) = f(x)$ at almost all $x \in [a,b]$. We then define $\int_a^b f = F(b) - F(a)$.

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Kubota integral

A function $f\colon [a,b] \to \mathbb{R}$ is said to be *Kubota integrable* if there exists an *approximately continuous* $[ACG]$-function $F\colon [a,b] \to \mathbb{R}$ such that $F'_{\text{ap}}(x) = f(x)$ at almost all $x \in [a,b]$. We then define $\int_a^b f = F(b) - F(a)$.

Wide Denjoy and Kubota Integrals

Wide Denjoy integral

A function $f\colon [a,b] \to \mathbb{R}$ is said to be *D-integrable* (*integrable in the wide Denjoy sense*) if there exists a *continuous* ACG-function $F\colon [a,b] \to \mathbb{R}$ such that $F'_{\text{ap}}(x) = f(x)$ at almost all $x \in [a,b]$. We then define $\int_a^b f = F(b) - F(a)$.

Kubota integral

A function $f\colon [a,b] \to \mathbb{R}$ is said to be *Kubota integrable* if there exists an *approximately continuous* $[ACG]$-function $F\colon [a,b] \to \mathbb{R}$ such that $F'_{\text{ap}}(x) = f(x)$ at almost all $x \in [a,b]$. We then define $\int_a^b f = F(b) - F(a)$.

It is easy to show that the Wide Denjoy integral strictly contains the (narrow) Denjoy integral and therefore the HK-integral. Thus the δ-fineness of the divisions in the definiton of ACG_δ is necessary.

Descriptive Characterizations of the Wide Denjoy and Kubota Integrals

An ACG-function on $[a, b]$ possesses an approximate derivative a.e. (see e.g., Saks) and so the class of continuous ACG-functions gives a descriptive charaterization of the Wide Denjoy integral, while the class of approximately continuous $[ACG]$-functions gives a descriptive charaterization of the Kubota integral.

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- Each ACG_r-function is an $[ACG]$-function.
- There exists a continuous ACG-function which is not L_r-differentiable on a set of positive measure for any r.
- There exists a function which is D-integrable but which is HK_r-integrable for no r.

Coefficients Problem for Trigonometric Series

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The following is a straightforward consequence of results found in J. C. Burkill (Proc. London Math. Soc., 1932 and 1951), corrected in H. Burkill (JMAA 1972):

Coefficients Problem for the CP-integral

If the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

converges to a CP-integrable function f almost everywhere on $[0, 2\pi]$ and the partial sums of (1) are bounded for each x except on a countable set, then

$$\begin{aligned} a_n &= \frac{1}{\pi} (CP) \int_0^{2\pi} f(x) \cos nx \, dx, \qquad n = 0, 1, 2, \ldots; \\ b_n &= \frac{1}{\pi} (CP) \int_0^{2\pi} f(x) \sin nx \, dx, \qquad n = 1, 2, \ldots \end{aligned} \tag{2}$$

Trigonometric Integrals

In order to strengthen this result it was necessary to show that any HK_r-integral function is integrable in the sense of Burkill's CP-integral.

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The mention of the gauge in the definition ACG_r proved to be inconvenient.

We went around this difficulty by defining a new class, the class of ACG_r^* functions, in which no mention of a gauge is made:

Trigonometric Integrals

ACG_r-functions (2004)

F is AC_r on E if for all $\varepsilon > 0$ there exist $\eta > 0$ and a gauge δ defined on $[a, b]$ such that for any δ-fine division $\{(I_i, x_i)\}_{i=1}^n$ in $[a, b]$ having tags in E such that $\sum_{i=1}^n |I_i| < \eta$ we have $\sum_{i=1}^n \left(\frac{1}{|I_i|} \int_{I_i} |F(y) - F(x_i)|^r dy \right)^{1/r} < \varepsilon$. F is an ACG_r-function on $[a, b]$ if $[a, b]$ can be written as countable union of sets on each of which F is an AC_r-function.

Trigonometric Integrals

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ACG_r^*-functions (2024)

F is AC_r^* on E if for all $\varepsilon > 0$ there exists $\eta > 0$ defined on $[a,b]$ such that for any division $\{(I_i, x_i)\}_{i=1}^n$ having tags in E such that $\sum_{i=1}^n |I_i| < \eta$ we have $\sum_{i=1}^n \left(\frac{1}{|I_i|} \int_{I_i} |F(y) - F(x_i)|^r dy \right)^{1/r} < \varepsilon$. F is an ACG_r^*-function on $[a,b]$ if $[a,b]$ can be written as countable union of sets on each of which F is an AC_r^*-function.

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ACG_r^*-functions (2024)

F is AC_r^* on E if for all $\varepsilon > 0$ there exists $\eta > 0$ defined on $[a, b]$ such that for any division $\{(I_i, x_i)\}_{i=1}^n$ having tags in E such that $\sum_{i=1}^n |I_i| < \eta$ we have $\sum_{i=1}^n \left(\frac{1}{|I_i|}\int_{I_i} |F(y) - F(x_i)|^r dy\right)^{1/r} < \varepsilon$. F is an ACG_r^*-function on $[a, b]$ if $[a, b]$ can be written as countable union of sets on each of which F is an AC_r^*-function.

Note that if one were to take away the η condition from the definition of ACG_r, one would once again obtain a class that fails to contain the function $F(x) = x$. The class ACG_r^* is NOT "belt and suspenders".

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A solution to the coefficients problem for HK_r

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PM, Skvortsov, Sworowski, Tulone (submitted to JMAA, 2024), proved:

- $ACG_r^* = ACG_r$
- $HK_r \subset CP$

A solution to the coefficients problem for HK_r

Since $HK_r \subset CP$ we have:

Coefficients Problem for the HK_r-integral

If the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{3}$$

converges to an ~~CP~~ HK_r-integrable function f almost everywhere on $[0, 2\pi]$ and the partial sums of (3) are bounded for each x except on a countable set, then

$$\begin{aligned} a_n &= \frac{1}{\pi} (\not{CP}\ HK_r) \int_0^{2\pi} f(x) \cos nx \, dx, \qquad n = 0, 1, 2, \ldots; \\ b_n &= \frac{1}{\pi} (\not{CP}\ HK_r) \int_0^{2\pi} f(x) \sin nx \, dx, \qquad n = 1, 2, \ldots \end{aligned} \tag{4}$$

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