Point-set games and functions with the hereditary small oscillation property

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Joint work with Marek Balcerzak and Piotr Szuca.

Bibliography

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Definitions

Assume that X is a metric space, Σ is non-empty family of non-empty subsets of X, and $f\colon X \to \mathbb{R}$. We say that f has:

- the continuous restriction property (CRP) with respect to Σ if
$$\exists_{P\in\Sigma}\ f|_P \text{ is continuous;}$$
- the hereditary continuous restriction property (HCRP) wrt Σ if
$$\forall_{P\in\Sigma}\ \exists_{Q\in\Sigma}\ Q\subseteq P\ \&\ f|_Q \text{ is continuous;}$$
- the hereditary small oscillation property (HSOP) wrt Σ if
$$\forall_{\varepsilon>0}\ \forall_{P\in\Sigma}\ \exists_{Q\in\Sigma}\ Q\subseteq P\ \&\ \operatorname{osc}(f|_Q, x) < \varepsilon \text{ for all } x\in Q.$$
- We have: $\mathrm{HCRP} \Rightarrow \mathrm{CRP}$ and $\mathrm{HCRP} \Rightarrow \mathrm{HSOP}$.

Motivations

Let X be a Polish space without isolated points and let $f\colon X \to \mathbb{R}$. Then:

- f is Marczewski measurable ((*s*)-measurable) iff it has the HCRP with respect to Perf.
- f is Baire measurable iff it has the HCRP with respect to the family G_{Res} of G_δ sets with the property: there is a nonempty open set U such that G is residual in U.
- f is $\overline{\mu}$-measurable with respect to the completion $\overline{\mu}$ of a finite nonatomic Borel measure μ on X iff it has the HCRP with respect to the family Perf^+ of all perfect subsets of X with positive measure.

J. B. Brown and H. Elalaoui-Talibi, *Marczewski-Burstin-like characterizations of σ-algebras, ideals, and measurable functions*, Colloq. Math. 82, 1999.

HSOP versus HCRP

Theorem

In all those examples the property HCRP *can be replaced by* HSOP*:*

- *f is Baire measurable iff it has the* HSOP *wrt G_{Res};*
- *f is $\overline{\mu}$-measurable iff it has the* HSOP *wrt* Perf^+*;*
- *f is Marczewski measurable iff it has the* HSOP *wrt* Perf*.*

The next example shows that HSOP and HCRP may be different.

Cliquish functions

Definition

$f\colon X \to \mathbb{R}$ is cliquish if for each non-empty open set W and $\varepsilon > 0$ there is a non-empty open set $U \subseteq W$ with $\mathrm{diam}(f(U)) < \varepsilon$.

- $f\colon X \to \mathbb{R}$ is cliquish iff it has the HSOP with respect to the family $\tau_0(X)$ of all non-empty open sets in X;
- there is $f_0\colon \mathbb{R} \to \mathbb{R}$ with the HSOP wrt $\tau_0(\mathbb{R})$ for which $\mathrm{int}(C(f_0)) = \emptyset$;
- if $f\colon \mathbb{R} \to \mathbb{R}$ has the HCRP wrt $\tau_0(X)$ then $\mathrm{int}(C(f)) \neq \emptyset$.
- Thus $f_0 \in \mathrm{HSOP} \setminus \mathrm{HCRP}$ (wrt $\tau_0(X)$).

Let X be a metric space, Σ is a non-empty family of non-empty subsets of X and $f\colon X \to \mathbb{R}$ is fixed. The game $G_{<\omega}(\Sigma, f)$ is defined as follows.

- At the step $k = 0$, Player I plays $P \in \Sigma$, then Player II plays a set $P_0 \in \Sigma$, $P_0 \subseteq P$.
- If $n \geq 1$, Player I plays a finite sequence $\langle x_i \colon k_{n-1} < i \leq k_n \rangle \subseteq P_{n-1}$, where $k_0 = 0$ and $k_{n-1} < k_n$, and Player II plays a $P_n \in \Sigma$, $P_n \subseteq P$:

Player I	P		$x_1, \ldots, x_{k_1}$		$x_{k_1+1}, \ldots, x_{k_2}$		$\cdots$	
Player II		P_0		P_1		P_2		$\cdots$

Player II wins the game if $\langle x_n \rangle$ is convergent and $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$. Otherwise, Player I wins.

The games $G_\lambda(\Sigma, f)$

- Let λ be an increasing sequence of natural numbers with $\lambda(0) = 0$. The definition of the game $G_\lambda(\Sigma, f)$ is the same as the definition of $G_{<\omega}(\Sigma, f)$ with the condition: $k_n = \lambda(n)$ for every $n \in \mathbb{N}$.
- Moreover, if $\exists_m \forall_n\ \lambda(n) = mn$, the game $G_\lambda(\Sigma, f)$ is denoted by $G_m(\Sigma, f)$, i.e. at the nth step of the game $G_m(\Sigma, f)$, Player I plays a finite sequence $\langle x_i \colon m(n-1) < i \leq mn \rangle$ in P_{n-1} and Player II plays a set $P_n \in \Sigma$ as follows:

Player I	P		$x_1, \ldots, x_m$		$x_{m+1}, \ldots, x_{2m}$		$\cdots$	
Player II		P_0		P_1		P_2		$\cdots$

- Other rules are as in $G_{<\omega}(\Sigma, f)$.

The game $G_1(\Sigma, f)$

The game $G_1(\Sigma, f)$: we assume that $\lambda(n) = n$ for every $n > 0$, i.e.

Player I	P		x_1		x_2		$\cdots$	
Player II		P_0		P_1		P_2		$\cdots$
- with the rules that for each integer $n \geq 0$:
  - $x_n \in P_{n-1}$;
  - $P \in \Sigma$ and $P_n \in \Sigma$ and $P_n \subseteq P$.

  Player II wins the game $G_1(\Sigma, f)$ if $\langle x_n \rangle$ is convergent and $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$. Otherwise, Player I wins.

Equivalence

We say that two games 1 and 2 are equivalent whenever each of players has a winning strategy in the game 1 if and only if he has a winning strategy in the game 2.

Theorem

Assume that Σ is a non-empty family of non-empty subsets of a metric space (X, d) and $f \in \mathbb{R}^X$. Then for every λ the games $G_\lambda(\Sigma, f)$ and $G_{<\omega}(\Sigma, f)$ are equivalent.

A family Σ is dense if, for each $P \in \Sigma$ and every ball $B(x, r)$ with $x \in P$ there exists $Q \in \Sigma$ contained in $P \cap B(x, r)$.

Theorem

Assume that (X, d) is a complete metric space and Σ is a dense family whose members are non-empty closed subsets of X. For every $f : X \to \mathbb{R}$:

1. *f has the HSOP wrt Σ iff Player II has a winning strategy in the game $G_1(\Sigma, f)$;*
2. *f has not the HSOP wrt Σ iff Player I has a winning strategy in the game $G_1(\Sigma, f)$.*

This means that the game $G_1(\Sigma, f)$ is determined.

Proof

We have to prove:

1. if f has HSOP wrt Σ then Player II has a winning strategy in $G_1(\Sigma, f)$;
2. if f has not HSOP wrt Σ then Player I has a winning strategy.

Proof: part 1

Assume that f has HSOP. Let's play $G_1(\Sigma, f)$.

- Player I: $P \in \Sigma$.
- Player II: $P_0 \in \Sigma$, $P_0 \subseteq P$ st $|f(x) - f(x')| < 1$ for all $x, x' \in P_0$.
- Player I: $x_1 \in P_0$, etc.

- At the kth move: Player I: $x_k \in P_{k-1}$,
  Player II: $P_k \in \Sigma$ st $P_k \subset P_{k-1} \cap B(x_k, \frac{1}{2^k})$,
  $|f(x) - f(x')| < \frac{1}{k}$ for $x, x' \in P_k$.

- After the game we obtain a Cauchy sequence $\langle x_n \rangle \subseteq X$ st $x_n \in P_k$ for $k \geq n$. Hence $\lim_n x_n = x \in \bigcap_{k=0}^{\infty} P_k$ and $|f(x) - f(x_k)| < \frac{1}{k}$, so $\lim_k(f(x_k)) = f(x)$.

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Proof: part 2

Assume that f has not HSOP. Then there are: $\varepsilon > 0$ and $P \in \Sigma$ st

$$\forall_{Q\in\Sigma, Q\subseteq P}\ \exists_{x\in Q}\ \mathrm{osc}(f|_Q, x) \geq \varepsilon.$$

Let's play $G_1(\Sigma, f)$.

- Player I: $P \in \Sigma$.
- Player II: $P_0 \in \Sigma$, $P_0 \subseteq P$.
- Player I: $x_1 \in P_0$, etc
- At the $k+1$ move: x_k, P_k are chosen. Player I fixes any $x \in P_k$ and considers 2 cases.
  1. if $|f(x) - f(x_k)| > \frac{\varepsilon}{4}$ then $x_{k+1} = x$.
  2. Let $|f(x) - f(x_k)| \leq \frac{\varepsilon}{4}$. Then $\exists_{x' \in P_k} |f(x') - f(x)| \geq \frac{\varepsilon}{2}$, so $|f(x') - f(x_k)| \geq \frac{\varepsilon}{4}$ and Player I picks $x_{k+1} = x'$.
- After the game we obtain a sequence $\langle x_n \rangle$ st $\langle f(x_k) \rangle$ is not convergent.

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- Player II: $P_0 \in \Sigma$, $P_0 \subseteq P$.
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  2. Let $|f(x) - f(x_k)| \leq \frac{\varepsilon}{4}$. Then $\exists_{x'\in P_k}\ |f(x') - f(x)| \geq \frac{\varepsilon}{4}$, so $|f(x') - f(x_k)| \geq \frac{\varepsilon}{4}$ and Player I picks $x_{k+1} = x'$.

- After the game we obtain a sequence $\langle x_n \rangle$ st $\langle f(x_k) \rangle$ is not convergent.

Corollaries

Assume that X is a Polish space without isolated points. Then:

- if f is (is not) measurable with respect to the completion of μ, then Player II (Player I) has a winning strategy in the game $G_1(\mathrm{Perf}^+, f)$;
- if f is (is not) Marczewski measurable, then Player II (Player I) has a winning strategy in the game $G_1(\mathrm{Perf}, f)$;
- if f is (is not) Baire measurable, then Player II (Player I) has a winning strategy in the game $G_1(G_{Res}, f)$;
- if f is (is not) cliquish, Player II (Player I) has a winning strategy in the game $G_1(\mathrm{CLO}, f)$.

Open problem

Problem

Do there exist a family Σ of non-empty subsets of $\mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ for which the game $G_1(\Sigma, f)$ is not determined?