On some open problems connected with stability and instability of certain properties of functions

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Dynamical systems in compact space X

Let $f\colon [0,1] \to [0,1]$ $(i = 1, 2, \ldots)$ be a function. Then

$$f^0 = \mathrm{id}_{[0,1]}, \;\; f^n = f \circ \ldots \circ f \;\; (n \text{ times})$$

A pair $([0,1], \{f^n\}_{n=1}^{\infty})$ is called **dynamical system** and it is denoted by (f).

Semigroup generated by family of function

Let Ψ be a family of functions. Put

$$S_n(\Psi) = \{f_{i_1} \circ \cdots \circ f_{i_n} : f_{i_1}, \ldots, f_{i_n} \in \Psi\}$$

for any $n \in N$. The set $S(\Psi) = \bigcup_{n=1}^{\infty} S_n(\Psi)$ is a semigroup of functions generated by the family Ψ. Then the family Ψ will be called the set of generators of the semigroup $S(\Psi)$.

Stable and unstable property

$\mathrm{id}(x) = x$, for $x \in [0, 1]$.
Let $\mathcal{P}$ be some property of function and let $\Psi \neq \{\mathrm{id}\}$ be a family of functions.

Stability

We will say that $\mathcal{P}$ is a **stable property for $\boldsymbol{\Psi}$** if each function $f \in S(\Psi) \setminus \{\mathrm{id}\}$ has the property $\mathcal{P}$.

Instability

We will say that $\mathcal{P}$ is an **unstable property for $\boldsymbol{\Psi}$** if it is not a stable property (i. e. there exists function $g \in S(\Psi) \setminus \{\mathrm{id}\}$ that does not have the property $\mathcal{P}$).

If Ψ is a family of all functions having property $\mathcal{P}$ then we will say briefly stable/unstable property.

QUASI-CONTINUITY

Quasi-continuous function

We will say that a function $f\colon [0,1] \to [0,1]$ is quasi-continuous at x_0 if for any open neighbourhoods W of $f(x_0)$ and U of x_0 there exists an open set $V \subset U$ such that $f(V) \subset W$.
A function $f\colon [0,1] \to [0,1]$ is quasi-continuous if it is quasi-continuous at each point of $[0,1]$.
The set of all quasi-continuous (Darboux and quasi-continuous) functions will be denoted Qc (DQc).

THEOREM [A. Peris], [M. Kucharska & RJP]

Quasi-continuity is an unstable property.

THEOREM [H. Pawlak & RJP], [M. Kucharska & RJP]

Darboux and quasi-continuity is a stable property .

TOPOLOGICAL ENTROPY

Let $\Psi = \{f_0 = \mathrm{id}, f_1, f_2, \ldots, f_k\}$, where $f_i \cdot [0,1] \to [0,1]$ for $i \in [\![0,k]\!]$, be the set of generators.

Entropy of semigroup of functions, A. Biś

Let $n \in N$, $\varepsilon > 0$ and $Y \subset [0,1]$. We say that the set $Z \subset Y$ is (n,ε)-separated by $S(\Psi)$ in Y if for any distinct points $p, q \in Z$ there exists function $g \in S_n(\Psi)$ such that $\mid g(p) - g(q) \mid > \varepsilon$. By $s_n(\varepsilon, S(\Psi), Y)$ we denote the maximal cardinality of the set (n,ε)-separated by $S(\Psi)$ in Y. Then the **entropy of semigroup** $S(\Psi)$ on the set Y is the number

$$h(S(\Psi), Y) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, S(\Psi), Y).$$

Entropy of function

Let $f \colon [0,1] \to [0,1]$. Then $h(f, Y) = h(S(f_0, f), Y)$

If $Y = [0,1]$, we will omit the symbol of set Y.

Instability

PROPOSITION

"Zero entropy" is an unstable property.

"Positive entropy" is an unstable property.

PROBLEMS

THEOREM [RJP]

For each $\alpha > 0$ there exists a finite family of continuous functions $\Psi = \{f_0 = \mathrm{id}, f_1, f_2, \dots, f_k\}$ ($f_i \colon [0,1] \to [0,1]$, $i \in [\![0,k]\!]$) such that property: *function has zero entropy* is stable for Ψ but $h(S(\Psi)) > \alpha$.

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PROBLEM

Can we assume $\alpha = \infty$ in the above theorem?

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PROBLEM

What kind of conditions should we impose on the finite family $\Psi = \{f_0 = \mathrm{id}, f_1, f_2, \ldots, f_k\}$ with the stable property: **function has zero entropy**, in order to have $h(S(\Psi)) = 0$.

Distribution function

B. Schweizer, J. Smítal, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*, Trans. Amer. Math. Soc. 344 (2), 1994, 737–754.

Let $x, y \in [0,1]$ and (f) be a dynamical system and $t > 0$.

$$\Phi_{x,y}^{(f)}(t) = \liminf_{n\to\infty} \frac{1}{n} \#(\{j \in [\![0, n-1]\!] \colon \mid f^j(x) - f^j(y)) \mid < t\})$$

Lower distribution function of x, y for f.

$$\Phi_{x,y}^{*\,(f)}(t) = \limsup_{n\to\infty} \frac{1}{n} \#(\{j \in [\![0, n-1]\!] \colon \mid f^j(x) - f^j(y)) \mid < t\})$$

Upper distribution function of x, y for f.

Distributionally chaotic system

Let $x, y \in X$. We shall say that a pair (x, y) is **distributionally chaotic** of type 1 (**D1** for short) for a dynamical system (f) (or function f) if $\Phi^{*(f)}_{x,y}(t) = 1$ for any $t > 0$ and there exists $t_0 > 0$ such that $\Phi^{(f)}_{x,y}(t_0) = 0$.

Distributionally chaotic system

Let $x, y \in X$. We shall say that a pair (x, y) is **distributionally chaotic** of type 1 (**D1** for short) for a dynamical system (f) (or function f) if $\Phi^{*(f)}_{x,y}(t) = 1$ for any $t > 0$ and there exists $t_0 > 0$ such that $\Phi^{(f)}_{x,y}(t_0) = 0$.

A set $A \subset X$ is called **distributionally scrambled set of type 1** (**DS-set** for brevity) for a dynamical system (f) (or function f) if $\#(A) > 1$ and for each $x, y \in A$ such that $x \neq y$ the pair (x, y) is D1 for this system.

A dynamical system (f) (or function f) is **distributionally chaotic** (**DC1** for brevity) of type 1 if there exists an uncountable DS-set for this system.

Instability of DC1

THEOREM [RJP]

DC1 is an unstable property.

Problem

What kind of conditions should we impose on the nonsingleton family Ψ consisting of DC1 functions in order to have:

$$\text{DC1 is a stable property for } \Psi?$$

Distributionally chaotic point

DC1 point

Let (f) be a dynamical system. We shall say that $x_0 \in X$ is a **DC1 point** (distributionally chaotic point) of (f) (or function f) if for any $\varepsilon > 0$ there exists an uncountable set S being a DS-set for (f) such that there are $n \in \mathbb{N}$ and a closed set $A \supset S$ fulfilling the condition

$$A \subset f^{i \cdot n}(A) \subset B(x_0, \varepsilon)$$

for $i \in \mathbb{N}$.

The set A described above will be called **(n, ε)-envelope** of the set S.

PROBLEMS

Theorem [On the local aspects of distributional chaos, Chaos 29 (2019), A. Loranty & RJP]

Let $f : [0,1] \to [0,1]$ be a continuous function such that $h(f) > 0$. Then there exists a point in $[0,1]$ which is a DC1 point of f (of the system (f)).

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Theorem [F. Balibrea (Spain) & L. Rucka (Czechia), December 2021]

Let $f : [0,1] \to [0,1]$ be a continuous function such that $h(f) > 0$. Then there exists an uncountable set of DC1-points of f (of the system (f)).

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Problem

What will happen if we weaken the assumption of continuity (e.g. Darboux, quasicontinuity; almost continuity; etc.)

0-approximately continuous functions

Let $\mathcal{L}$ denote the σ-algebra of all Lebesgue measurable sets and μ - the Lebesgue measure. For any $x_0 \in [0,1]$ and $A \in \mathcal{L}$ if there exists the limit

$$\lim_{h\to 0^+} \frac{\mu(A \cap [x_0 - h, x_0 + h])}{2h},$$

then we call it a *density of a set A at a point x_0* and denote it by $d(A, x_0)$. If $x_0 = 0$ or $x_0 = 1$ then we consider suitable one-sided density of this set at x_0. If $d(A, x_0) = 1$, then we say that x_0 is a *density point of a set A*.

0-approximately continuous functions

We shall say that a function f is **0-approximately continuous** at a point x_0 if there exists a set $A \in \mathcal{L}$ such that $d(A, x_0) = 1$ and $\lim_{A \ni x \to x_0} f(x) = f(x_0)$ and $h(f, A) = 0$.

Local instability

THEOREM [M. Kucharska & RJP]

Let $x_0 \in [0, 1]$, then each of the properties
- x_0 is a quasi-continuity point,
- x_0 is a 0-approximate continuity point,
- x_0 is a DC1 point

is unstable.

THEOREM

D($S(\Psi)$) is the set of all points x, such that x is a discontinuity point of any function $\xi \in S(\Psi)$.

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THEOREM

For each continuous function $f\colon [0,1] \to [0,1]$ and each $\varepsilon > 0$ there exist an uncountable family $\Psi \subset \mathrm{B}(f, \varepsilon)$ and a set $P \subset \mathrm{D}(S(\Psi))$ such that $\mu(P) > 0$ and:

1. *quasi-continuity* is a stable property for Ψ;
2. 0-*approximate continuity at each* $x \in P$ is a stable property for Ψ;
3. *each point* $x \in P$ *is a DC1 point* is a stable property for Ψ.

PROBLEM

What kind of assumptions should we impose on family Ψ in order to have:

there exists a set $P \subset \mathrm{D}(\mathcal{S}(\Psi))$ such that $\mu(P) > 0$ and:

1. *quasi-continuity* is a stable property for Ψ;
2. 0-*approximate continuity at each* $x \in P$ is a stable property for Ψ;
3. *each point* $x \in P$ *is a DC1 point* is a stable property for Ψ.

THANK YOU FOR YOUR ATTENTION

AND PATIENCE