Galois connection between regular subsets in topological space

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The 46th Summer Symposium in Real Analysis

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$$\begin{array}{ccccccc} & & \overline{\mathrm{Int(A)}} & \longrightarrow & \overline{\mathrm{Int}\overline{A}} & \longrightarrow & \overline{A} \\ & & \uparrow & & \uparrow & & \\ \mathrm{A} \longrightarrow & \mathrm{Int(A)} \longrightarrow & \mathrm{Int}\overline{\mathrm{Int(A)}} & \longrightarrow & \mathrm{Int}\overline{A} & & \end{array}$$

Classical types of subsets in topological space

The generalized closure operators (resp. interior operators) are determined by these operations and have the form $A \cup \Phi(A)$ (resp. $A \cap \Phi(A)$), where

$$\Phi(A) \in \{Int\overline{Int(A)}, Int\overline{A}, \overline{Int(A)}, Int\overline{A} \cap \overline{Int(A)}, Int\overline{A} \cup \overline{Int(A)}, \overline{Int\overline{A}}\}.$$

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Let's recall the classical types of the appropriate generalized open and generalized closed subsets:

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Let's recall the classical types of the appropriate generalized open and generalized closed subsets:

Definition

For a topological space (X, τ), we denote:

- $\mathscr{O}^\alpha := \{A \subset X : A \subset Int\overline{Int(A)}\}$, $\mathscr{C}^\alpha := \{A \subset X : \overline{Int\overline{A}} \subset A\}$
- $\mathscr{O}^s := \{A \subset X : A \subset \overline{Int(A)}\}$, $\mathscr{C}^s := \{A \subset X : Int\overline{A} \subset A\}$
- $\mathscr{O}^p := \{A \subset X : A \subset Int\overline{A}\}$, $\mathscr{C}^p := \{A \subset X : \overline{Int(A)} \subset A\}$
- $\mathscr{O}^\gamma := \{A \subset X : A \subset \overline{Int(A)} \cup Int\overline{A}\}$, $\mathscr{C}^\gamma := \{A \subset X : \overline{Int(A)} \cap Int\overline{A} \subset A\}$
- $\mathscr{O}^\beta := \{A \subset X : A \subset \overline{Int\overline{A}}\}$, $\mathscr{C}^\beta := \{A \subset X : Int\overline{Int(A)} \subset A\}$

Relationships among the families of subsets of classical types

As a consequence of Kuratowski's result, we have the following relationships among the families that we defined above

$$\begin{array}{ccccccccc} & & & & \mathscr{O}^{p} & & & & \\ & & & \nearrow & & \searrow & & & \\ \tau & \longrightarrow & \mathscr{O}^{\alpha} & & & & \mathscr{O}^{\gamma} & \longrightarrow & \mathscr{O}^{\beta} \\ & & & \searrow & & \nearrow & & & \\ & & & & \mathscr{O}^{s} & & & & \end{array}$$

$$\begin{array}{ccccccccc} & & & & \mathscr{C}^{p} & & & & \\ & & & \nearrow & & \searrow & & & \\ \mathscr{C} & \longrightarrow & \mathscr{C}^{\alpha} & & & & \mathscr{C}^{\gamma} & \longrightarrow & \mathscr{C}^{\beta} \\ & & & \searrow & & \nearrow & & & \\ & & & & \mathscr{C}^{s} & & & & \end{array}$$

The Kuratowski $\{b, i, \vee, \wedge\}$-problem.

Gardner, B. J-Jackson, M.G., The Kuratowski closure-complement theorem, New Zealand J.Math., (2008), 9-44.

Sherman, D., Variations on Kuratowski's 14-set theorem, The American Mathematical Monthly, (2010), 113-123

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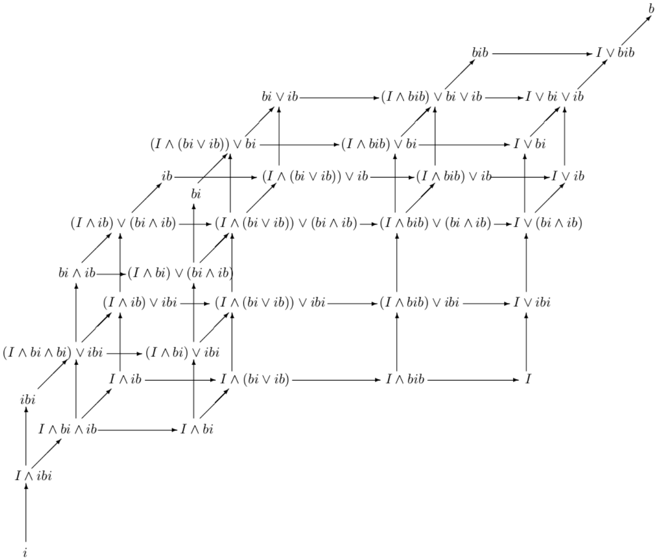
Question

What is the number of sets obtainable from a given subset of a topological space using the operators derived by composing members of the collection $\{b, i, \vee, \wedge\}$, where $\vee$ and $\wedge$ denote the union and intersection, respectively?

Hasse diagram



Operators from Hasse diagram

Each operator from this Hasse graph is a function

$$(\Phi, \Psi) : \mathscr{P}(X) \to \mathscr{P}(X)$$

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$$(\Phi, \Psi)(A) = (A \cap \Psi(A)) \cup \Phi(A)$$

or equivalently

$$(\Phi, \Psi)(A) = (A \cup \Phi(A)) \cap \Psi(A),$$

where $A \subset X$ and $\Phi, \Psi \in \{ibi, ib, bi, bi \wedge ib, bi \vee ib, bib\}$ i.e,
$\Phi, \Psi \in \{Int\overline{Int(\dots)}, Int\overline{(\dots)}, \overline{Int(\dots)}, Int\overline{(\dots)} \cap \overline{Int(\dots)}, Int\overline{(\dots)} \cup \overline{Int(\dots)}, \overline{Int\overline{(\dots)}}\}$

Fixed points

The problem of fixed points of such operators was examined in the paper "The lattices of families of regular sets in topological spaces." Mathematica Slovaca 70.2 (2020): 477-488.

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Definition of regular subsets:

As a result, for any pair (Φ, Ψ), the family of all fixed points that are characterized as follows

$$\mathscr{D}(\Phi, \Psi) = \{A \subset X : \Phi(A) \subset A \subset \Psi(A)\},$$

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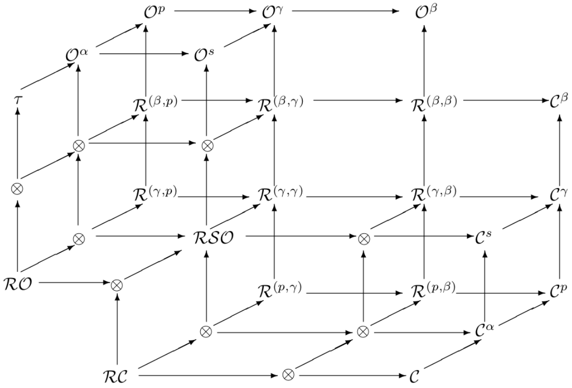
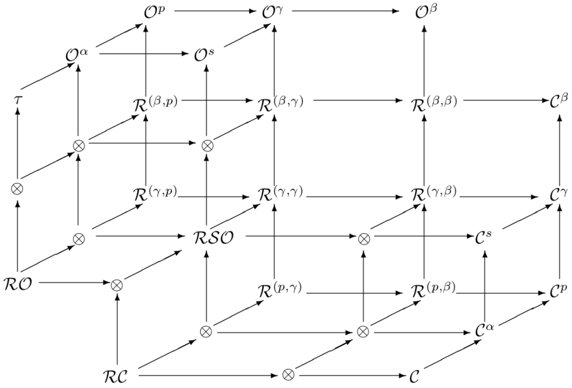
The elements of these families are called **regular subsets**.

The relationships among families of regular subsets





All families of regular subsets are of the form $\mathscr{R}^{(a,b)} = \mathscr{C}^a \cap \mathscr{O}^b$ where $\mathscr{C}^a \in \{\mathscr{C}, \mathscr{C}^\alpha, \mathscr{C}^s, \mathscr{C}^p . \mathscr{C}^\gamma, \mathscr{C}^\beta\}$ and $\mathscr{O}^b \in \{\tau, \mathscr{O}^\alpha, \mathscr{O}^s, \mathscr{O}^p . \mathscr{O}^\gamma, \mathscr{O}^\beta\}$.

The main result of this investigation is the following theorem.

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Theorem

Any family $\mathscr{D}(\Phi,\Psi)$ of regular subsets of a topological space (X,τ) has the structure of a complemented complete lattice under the operations defined as follows:

$$A \oplus B = (\Phi,\Psi)(A \cup B) = ((A \cup B) \cap \Psi(A \cup B)) \cup \Phi(A \cup B)$$

$$A \odot B = (\Phi,\Psi)(A \cap B) = ((A \cap B) \cap \Psi(A \cap B)) \cup \Phi(A \cap B)$$

$$A' = (\Phi,\Psi)(X \setminus A) = ((X \setminus A) \cap \Psi(X \setminus A)) \cup \Phi(X \setminus A)$$

Birkhoff, G., Lattice theory. Vol. 25. American Mathematical Soc., (1940).

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So, in our case in a family $\mathscr{D}(\Phi, \Psi)$ we have the lattice order $\prec$ given by

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It turns out that in every lattice $\mathscr{D}(\Phi,\Psi)$ the order means inclusion i.e.,

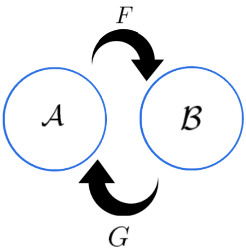
$$A \subset B \Leftrightarrow A \odot B = A \text{ and } A \oplus B = B,$$

- Ore, O., Galois connexions, Transactions of the American mathematical society (1944): 493-513.
- Birkhoff G., Lattice Theory, Amer. Math. Soc. Colloquium Publications, Providence, Rhode Island, 1st edition, 1940 (3rd edition 1967).
- Schmidt, J., Beitrage zur Filtertheorie. II, Mathematische Nachrichten 10.3 - 4 (1953): 197-232.

Definition

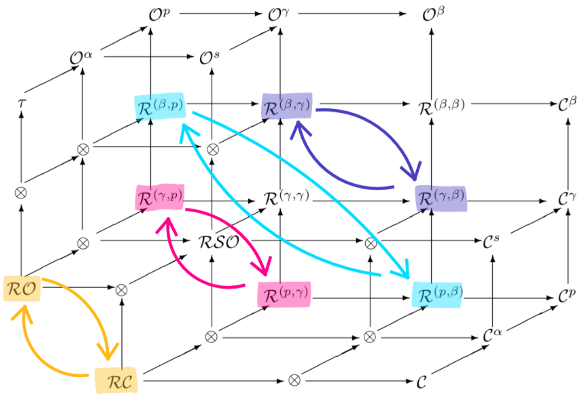
For the lattices $(\mathscr{A}, <_{\mathscr{A}}), (\mathscr{B}, <_{\mathscr{B}})$ and $F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{A}$, the pair (F,G) of functions is a Galois connection iff the following two clauses hold:

1. $x <_{\mathscr{A}} G(F(x))$ and $F(G(y)) <_{\mathscr{B}} y$
2. F and G are monotonic



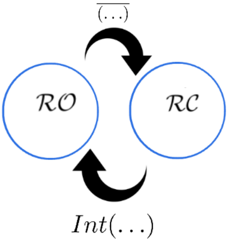
Question

Are there Galois connections between symmetric families of regular sets?



CASE I

Let us take the pair $(F, G) = (\overline{(\dots)}, Int(\dots))$. Then we get:



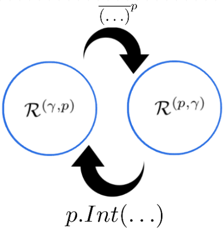
$$A \in \mathscr{RO} \xrightarrow{\overline{(\dots)}} \overline{A} \in \mathscr{RC} \xrightarrow{Int(\dots)} Int\overline{A} = A \in \mathscr{RO}$$

$$B \in \mathscr{RC} \xrightarrow{Int(\dots)} Int(A) \in \mathscr{RO} \xrightarrow{\overline{(\dots)}} \overline{Int(A)} = A \in \mathscr{RC}$$

So, we obtain $G(F(A)) = A$ and $F(G(B)) = B$ i.e., the pair $((\overline{(\dots)}, Int(\dots)))$, as it is usually called, is a perfect Galois connection.

CASE II

Using the pair of functions $(F,G) = (\overline{(\ldots)}^p, p.Int(\ldots))$ we obtain



$$A \in \mathscr{R}^{(\gamma,p)} \xrightarrow{\overline{(\ldots)}^p} \overline{A}^p \in \mathscr{R}^{(p,\gamma)} \xrightarrow{p.Int(\ldots)} \overline{A}^\gamma = A \in \mathscr{R}^{(\gamma,p}$$

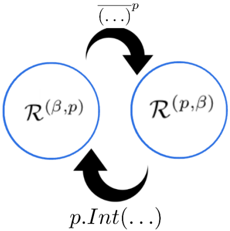
$$B \in \mathscr{R}^{(p,\gamma)} \xrightarrow{p.Int(\ldots)} p.Int(A) \in \mathscr{R}^{(\gamma,p)} \xrightarrow{\overline{(\ldots)}^p}$$
$$\xrightarrow{\overline{(\ldots)}^p} \gamma.Int(A) = A \in \mathscr{R}^{(p,\gamma)}$$

Hence, the pair $(\overline{(\ldots)}^p, p.Int(\ldots))$ is a perfect Galois connection either, because we have $G(F(A)) = A$ and $F(G(B)) = B$.

The case of the pair $(\mathscr{R}^{(\beta,p)}, \mathscr{R}^{(p,\beta)})$

CASE III

Taking the pair $(F,G) = (\overline{(\ldots)}^p, p.Int(\ldots))$ we obtain



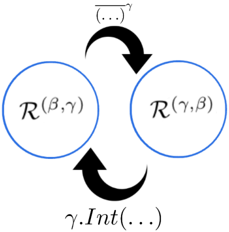
$$A \in \mathscr{R}^{(\beta,p)} \xrightarrow{\overline{(\ldots)}^p} \overline{A}^p \in \mathscr{R}^{(p,\beta)} \xrightarrow{p.Int(\ldots)} \overline{A}^\gamma \in \mathscr{R}^{(\beta,p)}$$

$$B \in \mathscr{R}^{(p,\beta)} \xrightarrow{p.Int(\ldots)} p.Int(A) \in \mathscr{R}^{(\beta,p)} \xrightarrow{\overline{(\ldots)}^p}$$
$$\xrightarrow{\overline{(\ldots)}^p} \gamma.Int(A) \in \mathscr{R}^{(p,\beta)}$$

Consequently, $A \subset G(F(A)) = \overline{A}^\gamma$ and $\gamma.Int(A) = F(G(B)) \subset B$, so we have the classical Galois connection.

CASE IV

Finally, let's take the pair $(F,G) = (\overline{(\ldots)}^{\gamma}, \gamma.Int(\ldots))$ and we have



$$A \in \mathscr{R}^{(\beta,\gamma)} \xrightarrow{\overline{(\ldots)}^{\gamma}} \overline{A}^{\gamma} \in \mathscr{R}^{(\gamma,\beta)} \xrightarrow{\gamma.Int(\ldots)} \overline{A}^{\gamma} \in \mathscr{R}^{(\beta,\gamma)}$$

$$B \in \mathscr{R}^{(\gamma,\beta)} \xrightarrow{\gamma.Int(\ldots)} \gamma.Int(A) \in \mathscr{R}^{(\beta,\gamma)} \xrightarrow{\overline{(\ldots)}^{\gamma}} \xrightarrow{\overline{(\ldots)}^{\gamma}} \gamma.Int(A) \in \mathscr{R}^{(\gamma,\beta)}$$

The conditions $A \subset G(F(A)) = \overline{A}^{\gamma}$ and $\gamma.Int(A) = F(G(B)) \subset B$ are satisfy, so we have the classical type of Galois connection.

- Ward, M., The closure operators of a lattice, Annals of Mathematics (1942): 191-196.
- Erne, M., A primer on Galois connections, Annals of the New York Academy of Sciences (1993): 103-125.

Definition

Given a lattice $(\mathscr{L}, <)$, a function $C : L \to L$ is called a closure operator (resp. interior operator) in $\mathscr{L}$ iff for every $x \in L$:

1. $x < C(x)$ (resp. $C(x) < x$)
2. $x < y$ implies $C(x) < C(y)$
3. $C(C(x)) = C(x)$

Closure and interior operator

For a given pair of lattices $(\mathscr{A}, <_{\mathscr{A}}), (\mathscr{B}, <_{\mathscr{B}})$ and a pair of functions $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{A}$ one of the classical result says as follows

For a given pair of lattices $(\mathscr{A}, <_{\mathscr{A}}), (\mathscr{B}, <_{\mathscr{B}})$ and a pair of functions $F : \mathscr{A} \to \mathscr{B}$, $G : \mathscr{B} \to \mathscr{A}$ one of the classical result says as follows

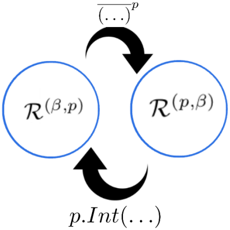
Lemma

If (F, G) is a Galois connection, then:

1. $A \to G(F(A))$ *is a closure operator in* $(\mathscr{A}, <_{\mathscr{A}})$*, where* $A \in \mathscr{A}$.
2. $B \to F(G(B))$ *is an interior operator in* $(\mathscr{B}, <_{\mathscr{B}})$*, where* $B \in \mathscr{B}$.

Conclusion

For the pair $(\mathscr{R}^{(\beta,p)}, \mathscr{R}^{(p,\beta)})$ or $(\mathscr{R}^{(\beta,\gamma)}, \mathscr{R}^{(\gamma,\beta)})$



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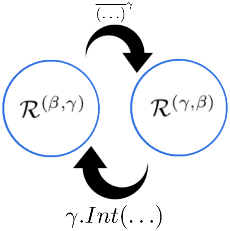

According to the cases III and IV, we have the following result:

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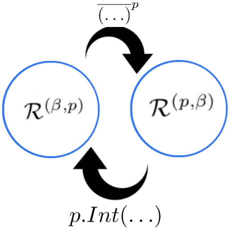

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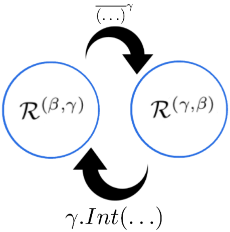

According to the cases III and IV, we have the following result:

Theorem

1. *The function* $A \to \overline{A}^\gamma$ *is a closure operator in* $\mathscr{R}^{(\beta,p)}$ *and* $\mathscr{R}^{(\beta,\gamma)}$,
2. *The function* $B \to \gamma.Int(B)$ *is an interior operator in* $\mathscr{R}^{(p,\beta)}$ *and* $\mathscr{R}^{(\gamma,\beta)}$.

Frink O., Topology in lattices, Transactions of the American Mathematical Society, 51 (1942): 569-582.

Frink has defined the interval topology of a lattice $(\mathscr{L}, \prec)$ by taking as a sub-basis for the closed sets all closed intervals $[a,b]$, $(-\infty, a]$ and $[b, \infty)$, where $[a,b] = \{x \in \mathscr{L} : a \prec x \prec b\}$, $(-\infty, a] = \{x \in \mathscr{L} : x \prec a\}$, $[b, \infty) = \{x \in \mathscr{L} : a \prec b\}$.

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In the lattices of type $\mathscr{R}^{(i,j)}$, where $(i,j) \in \{(\beta,p),(p,\beta),(p,\gamma),(\gamma,p),(\beta,\gamma),(\gamma,\beta)\}$ the intervals have the following form:

$$(A,B) = \{K \in \mathscr{R}^{(i,j)} : A \subset K \subset B\}$$

The definitions of the families of regular subsets i.e.,

$$\mathscr{D}(\Phi,\Psi) = \{A \subset X : \Phi(A) \subset A \subset \Psi(A)\}$$

suggest that the intervals of form of type $(\Phi(A), \Psi(A))$ play a special role in such families.

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Let's consider this issue on the example of one of these families, namely $\mathscr{R}^{(\beta,p)} = \{A \subset X : Int\overline{Int(A)} \subset A \subset Int\overline{A}\}$.

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$$(Int\overline{Int(A)}, Int\overline{A}) = \{K \in \mathscr{R}^{(\beta,p)} : Int\overline{Int(A)} \subset K \subset Int\overline{A}\},$$

where $A \in \mathscr{R}^{(\beta,p)}$.

Property I

For every $A, B \in \mathscr{R}^{(\beta,p)}$ the following properties are equivalent:

1. $B \in (Int\overline{Int(A)}, Int\overline{A})$,
2. $(Int\overline{Int(B)}, Int\overline{B}) \subset (Int\overline{Int(A)}, Int\overline{A})$.

Property II

The family $\mathscr{B}^{(\beta,p)} = \{(Int\overline{Int(A)}, Int\overline{A}) : A \in \mathscr{R}^{(\beta,p)}\}$ is a cover of $\mathscr{R}^{(\beta,p)}$.

So, it is clear that $\mathscr{B}^{(\beta,p)}$ is a base for some topology $\mathscr{T}^{(\beta,p)}$.

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What are the properties of the topology $\mathscr{T}^{(\beta,p)}$ *?*

Theorem

$(\mathscr{B}^{(\beta,p)}, \mathscr{T}^{(\beta,p)})$ *is an Aleksandroff topological space.*

Specialization order and the Aleksandroff topology

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In our investigation the specialization preorder is define as follows

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$$a < b \Leftrightarrow \{a\} \in \overline{\{b\}},$$

which is reflexive and transitive, called the specialization preorder.

In our investigation the specialization preorder is define as follows

$$A < B \Leftrightarrow (Int\overline{Int(A)}, Int\overline{A}) \supset (Int\overline{Int(B)}, Int\overline{B})$$

Relationship between $\mathscr{T}^{(\beta,p)}$ and the interval topology.

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Remark

The topology $\mathscr{T}^{(\beta,p)}$ is not T_1.

Alexandroff P., Diskrete raume, 2(3) (1937) , 501-519.

Classical results says that

Lemma

An Aleksandroff space is T_0 if and only if the equality of minimal open neibourhoods of points implies the equality of this points.

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Example

$A = (0,1) \cup ([1,2) \cap \mathbb{Q}), B = (0,1) \cup ([1,2) \cap \mathbb{IQ})$

Conclusion

- The space $(\mathscr{B}^{(\beta,p)}, \mathscr{T}^{(\beta,p)})$ is an Aleksandroff topological space which is not T_0.

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- The space $(\mathscr{B}^{(\beta,p)}, \mathscr{T}^{(\beta,p)})$ is an Aleksandroff topological space which is not T_0.

  Consequently,
- Topology $(\mathscr{B}^{(\beta,p)}, \mathscr{T}^{(\beta,p)})$ is different than the interval topology.

Thank you for your attention!!!