ON HK$_r$- AND P$_r$-INTEGRABLE FUNCTIONS

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L^r-DERIVATIVE

Let $\infty > r \geq 1$.

A function $F \in L^r[a,b]$ is said to be L^r-differentiable at $x \in (a,b)$, if there exists a real number α such that

$$\left(\frac{1}{h}\int_{-h}^{h} |F(x+t) - F(x) - \alpha t|^r \, dt\right)^{1/r} = o(h)$$

as $h \to 0$. In this case we say that α is the L^r-*derivative* of F at x and write $F'_r(x) = \alpha$.

A function $F \in L^r[a,b]$ is said to be L^r-continuous at $x \in (a,b)$, if

$$\left(\frac{1}{h}\int_{-h}^{h} |F(x+t) - F(x)|^r \, dt\right)^{1/r} = o(1)$$

as $h \to 0$.

L^r-DINI DERIVATES

The upper righthand L^r-derivate of F at $x \in [a, b)$, denoted by $D_r^+ F(x)$, is defined as the infimum of all numbers α such that

$$\left(\frac{1}{h} \int_0^h [F(x+t) - F(x) - \alpha t]_+^r \, dt \right)^{1/r} = o(h) \tag{1}$$

as $h \to 0^+$. If no real number α satisfies (1), we set $D_r^+ F(x) = +\infty$. The lower righthand L^r-derivate of F at x is

$$D_{+,r} F(x) = -D_r^+(-F)(x)$$

The upper and lower lefthand L^r-derivates at $x \in (a, b]$, as well as bilateral L^r-derivates are understood accordingly.

L^r-PERRON INTEGRAL (P_r-INTEGRAL)

Louis Gordon 1966

Let $f\colon [a,b] \to \mathbb{R}$.

A function $m \in L^r[a,b]$ is said to be an L^r-minor function of $f\colon [a,b] \to \mathbb{R}$ if

- $m(a) = 0$,
- m is L^r-continuous everywhere in $[a,b]$,
- $\overline{D}_r m(x) \leq f(x)$ at almost every $x \in [a,b]$,
- $\overline{D}_r m(x) < \infty$ nearly everywhere in $[a,b]$.

We say M is an L^r-major function for f if $-M$ is an L^r-minor function for $-f$.

L^r-PERRON INTEGRAL (P_r-INTEGRAL)

Louis Gordon 1966

Let $f\colon [a,b] \to \mathbb{R}$.

An f is said to be L^r-Perron integrable (or P_r-integrable) if for every $\varepsilon > 0$ there are an L^r-major function M and an L^r-minor function for f such that

$$M(b) - m(b) < \varepsilon.$$

We define

$$\int_a^b f = \sup_m m(b) = \inf_M M(b).$$

LEMMA

If $M, m\colon [a,b] \to \mathbb{R}$ are respectively an L^r-major and L^r-minor functions for f, then $M - m$ is nondecreasing.

A division in a segment $[a, b]$ is a finite family $\{(I_i, x_i)\}_{i=1}^k$, where the segments $I_i \ni x_i$ pairwise do not overlap; it is called a partition of $[a, b]$ if, moreover, $\bigcup_{i=1}^k I_i = [a, b]$.

Let $\delta\colon A \to (0, \infty)$ (a gauge), $A \subset \mathbb{R}$. We say that a pair (I, x) is δ-fine, if

$$I \subset (x - \delta(x), x + \delta(x)).$$

A division $\{(I_i, x_i)\}_{i=1}^k$ of $[a, b]$ is said to be δ-fine, if all (I_i, x_i) are such.

LEMMA ***Cousin 1895***

For every $[a, b]$ and any gauge δ on $[a, b]$ there is a δ-fine partition of $[a, b]$.

L_r-KURZWEIL–HENSTOCK INTEGRAL Paul Musial, Yoram Sagher 2004

PAUL MUSIAL

Let $\infty > r \geq 1$.

DEFINITION

A function $f\colon [a,b] \to \mathbb{R}$ is L^r-Kurzweil–Henstock integrable (KH_r-integrable) on $[a,b]$ if there exists a function $F \in L^r$ such that for any $\varepsilon > 0$ there exists a gauge δ such that for any δ-fine partition $\{(I_i, x_i)\}_{i=1}^{k}$ of $[a,b]$ we have

$$\sum_{i=1}^{k}\left(\frac{1}{|I_i|}\int_{I_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r} < \varepsilon.$$

We set then

$$\int_a^b f = F(b) - F(a).$$

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RELATIONS

THEOREM **Paul Musial, Yoram Sagher 2004**
If a function $f\colon [a,b] \to \mathbb{R}$ is P_r-integrable, then it is HK_r-integrable with the same value of the integral.

$$\mathrm{P} = \mathrm{HK} \;\Rightarrow\; \mathrm{P}_r\,.$$

$$\mathrm{P}_r,\; r > s \;\Rightarrow\; \mathrm{P}_s\,.$$

$$\mathrm{HK}_r,\; r > s \;\Rightarrow\; \mathrm{HK}_s\,.$$

The implications, obviously, cannot be reversed.

A SURPRISING RESULT (2022)

 Valentin SKVORTSOV

Francesco TULONE 

THEOREM
Given r, there is an HK_r-integrable function which is not P_r-integrable!

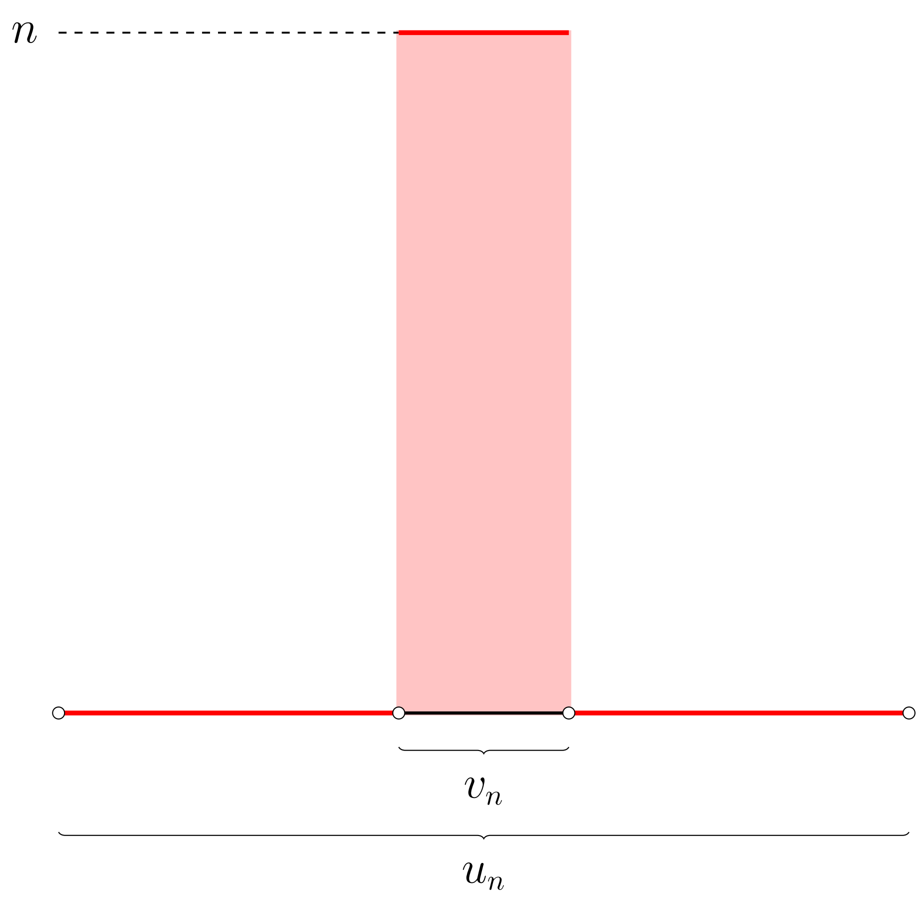
Musial, P., Skvortsov, V., Tulone, F.: The HK_r-integral is not contained in the P_r-integral. Proc. Amer. Math. Soc. **150**(5), 2107–2114 (2022)

AN ENHANCEMENT (2024)

THEOREM
Given r, there is an HK_r-integrable function which is P_s-integrable for no s!

Sworowski, P.: *An HK_r-integrable function which is P_s-integrable for no s*, arXiv:2403.11733 (2024). submitted

CONSTRUCTION



$$u_n = \frac{4^r - 2}{4^{rn}}$$

$$v_n = \frac{u_n}{3^{nr}}$$

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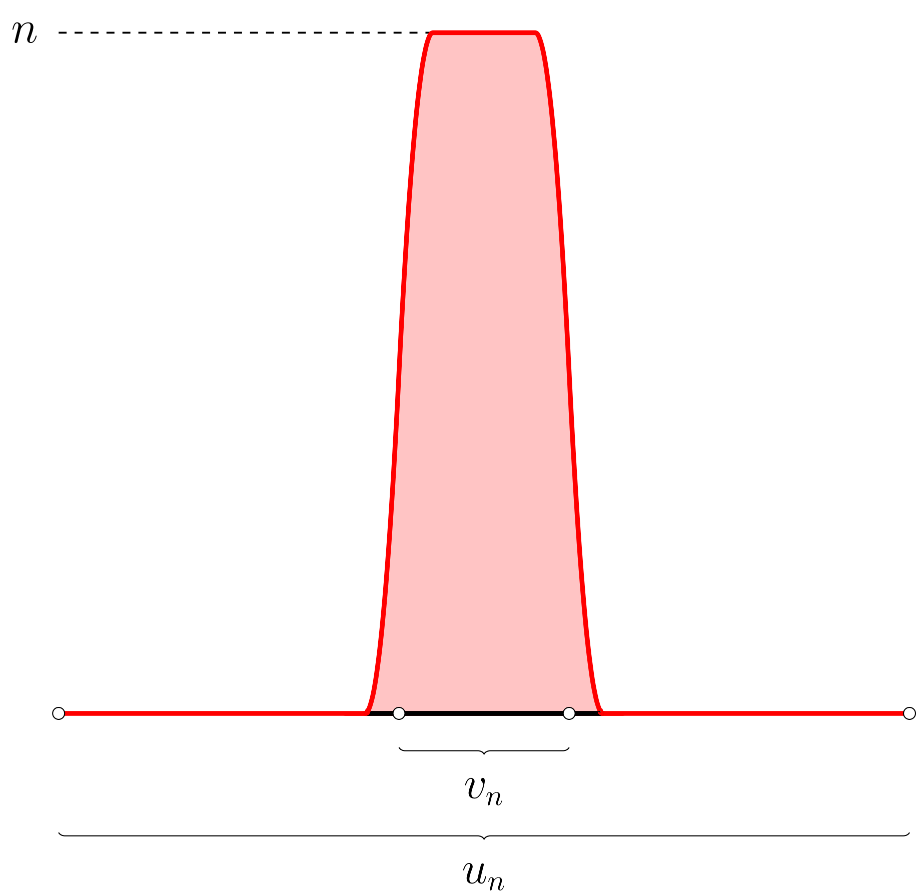
CONSTRUCTION



$$u_n = \frac{4^r - 2}{4^{rn}}$$

$$v_n = \frac{u_n}{3^{nr}}$$

A proof that F' is not P_1-integrable

Suppose not, then $F = (\mathrm{P}_1) \int F'$.

m — an L_1-minor function m for F'

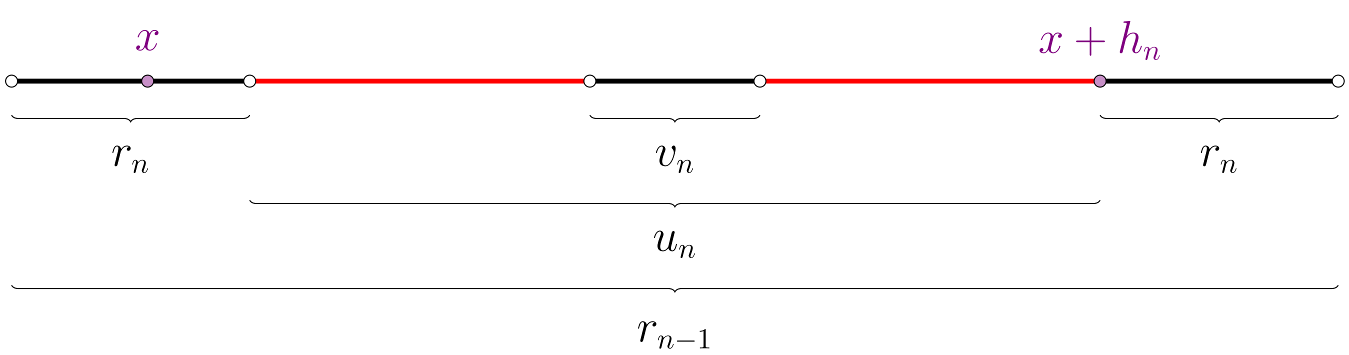
$R = F - m$ is non-decreasing on $[0, 1]$

$x \in P$

N — arbitrarily large, take an $n > N$ such that

$$x \in r_n \subset r_{n-1}.$$

$x + h_n$ — the right endpoint of u_n



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One has

$$r_n < u_n, \quad \text{so} \quad h_n < r_n + u_n < 2u_n.$$

$\alpha \in \mathbb{R}$ — arbitrary and consider

$$n > R(1) + |\alpha| + 1$$

then

$$[m(x+t) - m(x) - \alpha t]_+ = [(F(x+t) - R(x+t) + R(x) - \alpha t]_+ > 1$$

for all $x + t \in v_n \subset u_n$.

Estimate

$$\frac{1}{h_n^2} \int_0^{h_n} [m(x+t) - m(x) - \alpha t]_+ \, dt > \frac{v_n}{h_n^2} > \frac{v_n}{4u_n^2} = \frac{1}{4(2^{2r} - 2)} \cdot \left(\frac{4}{3}\right)^{nr}.$$

This means,

$$\overline{D}_r m(x) = \infty.$$

x here comes from a co-countable subset of P, so m cannot be an L^1-minor function for F'. ♦

REMARK

The so-constructed F is an indefinite HK_s-integral iff $s < r \log_2 3$.

PROBLEM

Is there a function which is not P_1-integrable while HK_r-integrable for all r?