Algebrability and symmetric properties

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Symmetric continuity

Definition (Hausdorff, 1935)

We say that $f:\mathbb{R}\to\mathbb{R}$ *is symmetrically continuous at a point* $x\in\mathbb{R}$ *if*

$$\lim_{h\to 0}\big[f(x+h)-f(x-h)\big]=0.$$

The family of functions which are symmetrically continuous at any point will be denoted by $\mathcal{SC}$.

First results:

H. Fried, *Über die symmetrische Steigkeit von Funktionen*, Fund. Math. 29 (1937), 134–137.

Some properties

1. Any symmetrically continuous function is measurable.

I.N. Pesin, *On the measurability of symmetrically continuous functions*, Teor. Funkcii Funkktsional. Anal. i Prilozen 5 (1967), 99–101.

D. Preiss, *A note on symmetrically continuous functions*, Časopis Pěst. Mat. 96 (1971), 262–264.

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2. Stein and Zygmund proved that the set of points at which a symmetrically continuous function is not continuous is of measure zero and of first category.

   E.M. Stein, A. Zygmund, *On the differentiability of functions*, Studia Math. 23 (1960), 295–307.

Some properties

3. The set of points at which a symmetrically continuous function is not continuous may be uncountable.

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   D. Preiss, *A note on symmetrically continuous functions*, Časopis Pěst. Mat. 96 (1971), 262–264.

4. The above set does not contain a ternary Cantor set.

   M. Repický, *Sets of points of symmetric continuity*, Arch. Math. Logic (2015) 54:803–824.

Symmetric differentiability

Definition

We say that $f : \mathbb{R} \to \mathbb{R}$ *is symmetrically differentiable at a point* $x \in \mathbb{R}$ *if the limit*

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists and is finite. This value is called the symmetric derivative of f *at a point* x *and will be denoted by* $f^s(x)$*.*

The family of functions which are symmetrically differentiable at any point will be denoted by $\mathcal{SD}$.

A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fund. Math. 9 (1927), 212–279.

Z. Charzyński, *Sur lest fonctions dont la deriveé symetrique est partout finie*, Fund. Math. 21 (1931), 214–225.

Some properties

1. In 1923 Steinhhaus asked a question whether there is nonmeasurable function f whose symmetric derivative everywhere is equal to zero. The answer was negative.

Theorem (Charzyński, 1931)

The set of discontinuity points of everywhere symmetrically differentiable function is scattered.

H. Steinhaus, *Probl'eme #23*, Fund. Math., p. 368, 1923.

Z. Charzyński, *Sur lest fonctions dont la deriveé symetrique est partout finie*, Fund. Math. 21 (1931), 214–225.

Some properties

2. If a function f has both one-sided derivatives $f'(x_0^-)$ and $f'(x_0^+)$ then it has a symmetric derivative at x_0. Moreover, $f^s(x_0) = \frac{1}{2}\big(f'(x_0^-) + f'(x_0^+)\big)$.

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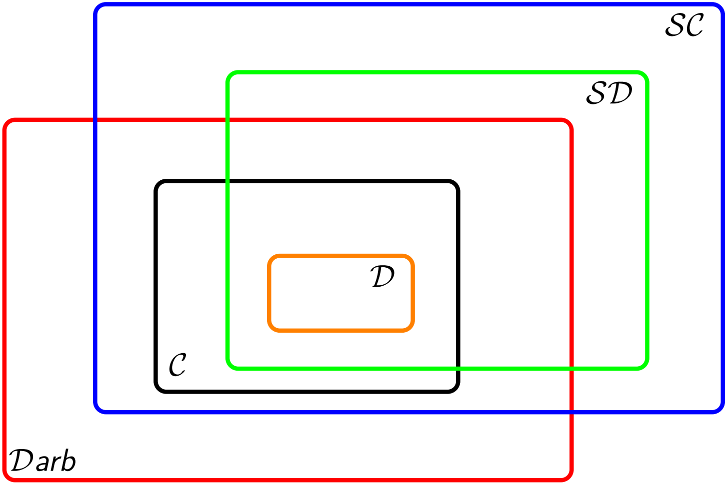
3. A symmetric derivative may not have the Darboux property contrary to an ordinary derivative which always has the Darboux property.
4. $$\mathcal{SD} \subsetneq \mathcal{SC} \subsetneq \mathcal{L} \quad \text{and} \quad \mathcal{D} \subsetneq \mathcal{SD} \not\subset \mathcal{D}arb$$



$\mathcal{SC}$
$\mathcal{SD}$
$\mathcal{D}$
$\mathcal{C}$
$\mathcal{D}arb$

Algebrability

Let κ be a cardinal number.

Definition

We will say that a subset A of a commutative algebra is κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra B, i.e. the minimal cardinality of a set of generators of B is equal to κ.

Definition

We say that A is strongly κ-algebrable if $A \cup \{0\}$ contains a κ-generated algebra B that is isomorphic with the free algebra.

- R.M. Aron, J.B. Seoane-Sepulveda, *Algebrability of the set of everywhere surjective functions on* $\mathbb{C}$, Bulletin of the Belgian Mathematical Society-Simon Stevin 14.1 (2007), 25–31.
- A. Bartoszewicz, S. Głąb, *Strong algebrability of sets of sequences and functions*, Proc. Amer. Math. Soc. 141.3 (2013), 827–835.

Do the families algebrable or strongly algebrable and what is the level of such algebrability?

- $\mathcal{SC}$
- $\mathcal{SC} \setminus \mathcal{C}$
- $\mathcal{SC} \setminus \mathcal{SD}$
- $\mathcal{D}arb \setminus \mathcal{SC}$
- $\mathcal{D}arb \setminus \mathcal{SD}$
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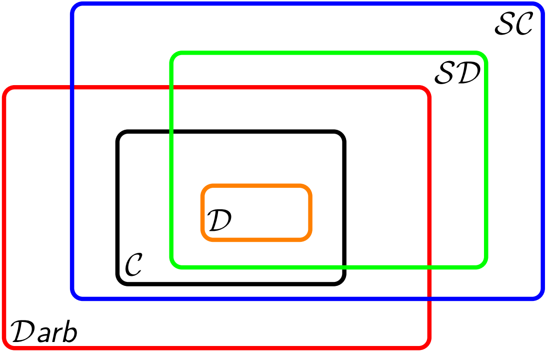

Theorem (Chlebik, 1991)

$\text{card}(\mathcal{SC}) = 2^{\mathfrak{c}}$.

M. Chlebik, *There are $2^{\mathfrak{c}}$ symmetrically continuous functions*, Proc. Amer. Math. Soc. 113 (1991), 683–688.

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Theorem (BFT, 2024)

$\mathrm{card}(\mathcal{SD}) = \mathfrak{c}$.

If $f, g \in \mathcal{SC}$ and $k \in \mathbb{R}$, then $f \pm g$, kf, $|f|$, $\max\{f, g\}$, $\min\{f, g\}$ are symmetrically continuous.
Moreover if $f, g \in \mathcal{SC}$ are locally bounded then $fg \in \mathcal{SC}$.
Hence the family of locally bounded symmetrically continuous functions is the algebra contained in $\mathcal{SC}$.

Example

Take $f(x) = \begin{cases} \frac{1}{x^2} & \text{for } x \neq 0 \\ a & \text{for } x = 0 \end{cases}$ *(*$a \in \mathbb{R}$ *is arbitrary) and* $g(x) = x$.
Then f *is not locally bounded,* $f, g \in \mathcal{SC}$ *and* $fg \notin \mathcal{SC}$.

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Then f is not locally bounded, $f, g \in \mathcal{SC}$ and $fg \notin \mathcal{SC}$.

Corollary

The family $\mathcal{SC}$ is $2^{\mathfrak{c}}$-algebrable.

Theorem (Fichtenholz-Kantorovich)

For any set X there exists a family $\mathcal{N} \subset \mathcal{P}(X)$ of cardinality $2^{|X|}$ such that for any finite sequences $N_1, \ldots, N_n \in \mathcal{N}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$

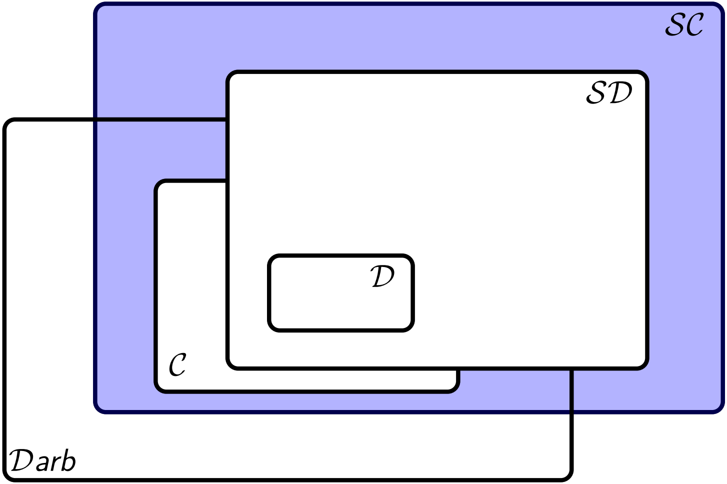
$$N_1^{\varepsilon_1} \cap \ldots \cap N_n^{\varepsilon_n} \neq \varnothing$$

(where $N^1 = N$ and $N^0 = X \setminus N$). Moreover all the sets in $\mathcal{N}$ can be chosen with the cardinality $|X|$. Such a family $\mathcal{N}$ is called an independent one.

G.M. Fichtenholz, L.V. Kantorovich, *Sur le opérations linéares dans l'espace de fonctions bornées*, Studia Math. 5 (1934), 69–98.

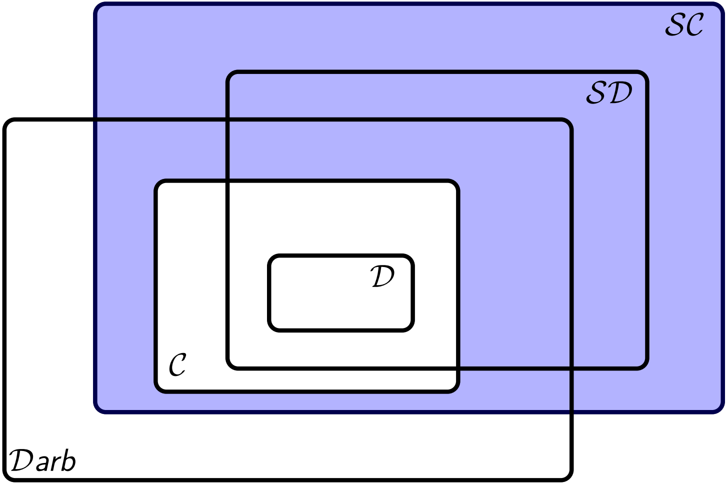
Theorem

The family $\mathcal{SC} \setminus (\mathcal{SD} \cup \mathcal{C})$ is $2^{\mathfrak{c}}$-algebrable.

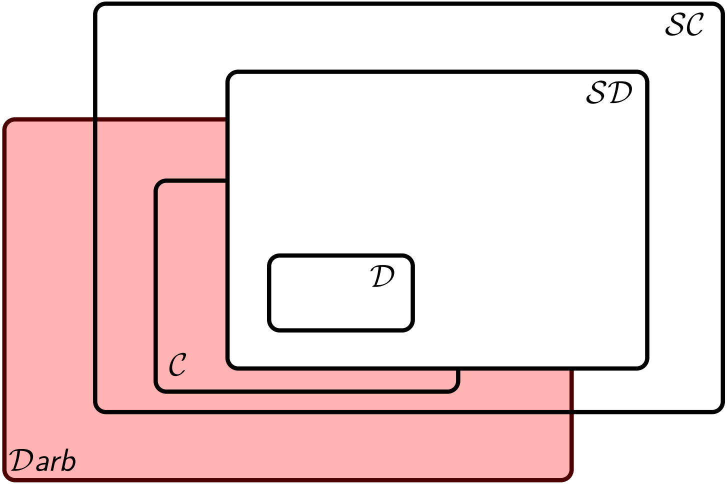


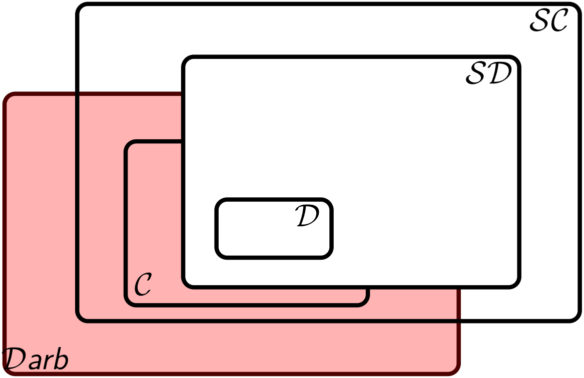
Corollary

The family $\mathcal{SC} \setminus \mathcal{C}$ is $2^{\mathfrak{c}}$-algebrable.



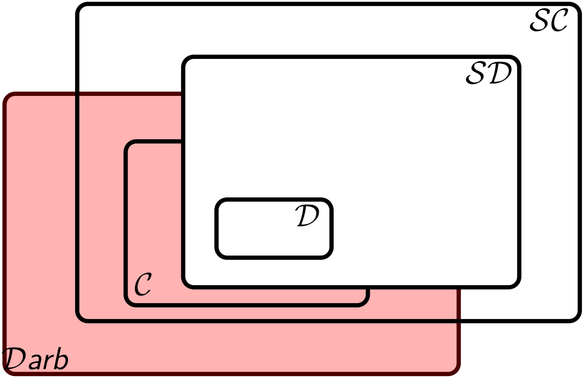
Is the family $\mathcal{D}arb \setminus \mathcal{SD}$ algebrable?





Theorem

The family $\mathcal{L} \cap \mathcal{B} \cap \mathcal{D}arb \setminus \mathcal{SC}$ *is strongly* $2^{\mathfrak{c}}$*-algebrable.*



Theorem

The family $\mathcal{L} \cap \mathcal{B} \cap \mathcal{D}arb \setminus \mathcal{SC}$ *is strongly* $2^{\mathfrak{c}}$*-algebrable.*

Corollary

$\mathcal{D}arb \setminus \mathcal{SD}$ *is strongly* $2^{\mathfrak{c}}$*-algebrable.*

Definition

We say that a function $f : \mathbb{R} \to \mathbb{R}$ *is exponential-like of a rank* m *if it is given by* $f(x) = \sum_{i=1}^{m} a_i e^{\beta_i x}$ *for some distinct nonzero numbers* $\beta_1, \ldots, \beta_m$ *and some nonzero real numbers* $a_1, \ldots, a_m$.

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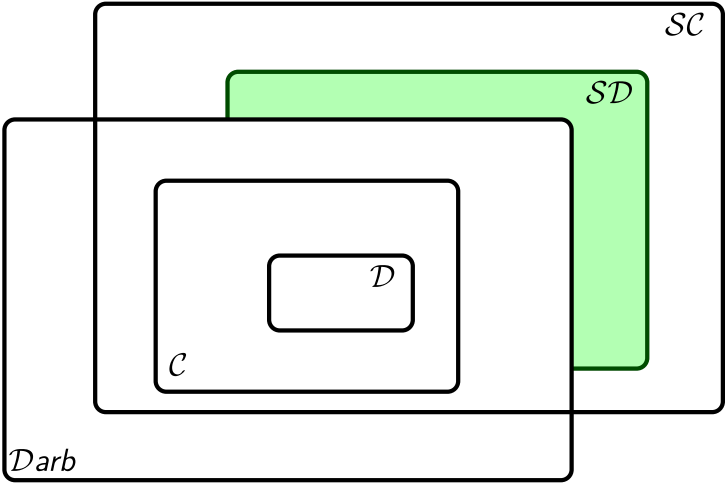
Theorem

Given a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, assume that there exists a function $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{0\}$ for every exponential-like function $f : \mathbb{R} \to \mathbb{R}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More exactly, if $H \subset \mathbb{R}$ is a set of cardinality $\mathfrak{c}$, linearly independent over the rationals $\mathbb{Q}$, then $\exp \circ (rF)$, $r \in H$, are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$.

M. Balcerzak, A. Bartoszewicz, M. Filipczak, *Nonseparable spaceability and strong algebrability of sets of continuous singular functions*, J. Math. Anal. Appl. 407.2 (2013), 263–269.

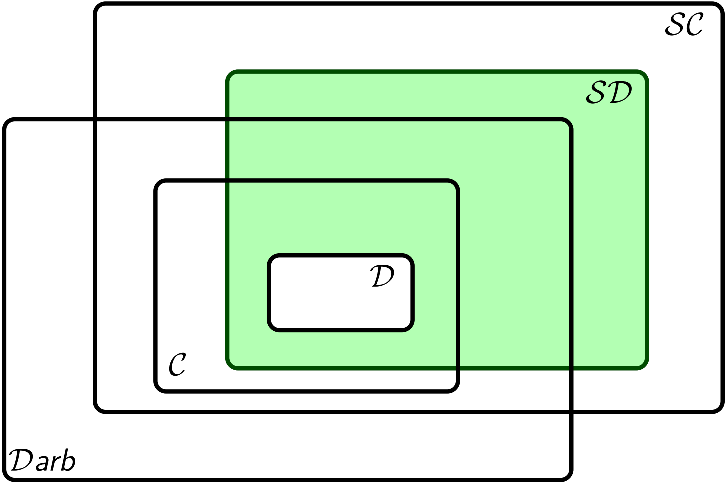
Theorem

The family $\mathcal{SD} \setminus \mathcal{D}arb$ is strongly $\mathfrak{c}$-algebrable.



Corollary

The family $\mathcal{SD} \setminus \mathcal{D}$ is strongly $\mathfrak{c}$-algebrable.



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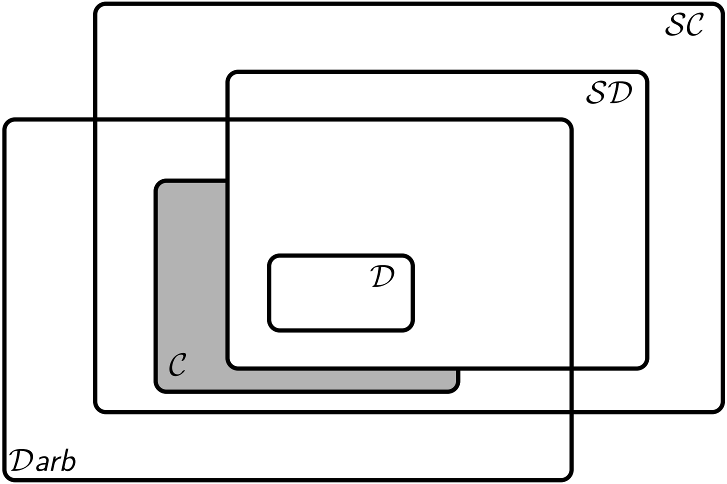
Theorem

There exists a function $F\colon [0,1] \to \mathbb{R}$ *which is continuous everywhere and does not have a symmetric derivative at any point.*

F.M. Filipczak, *Sur la structure de l'ensemble des points où une fonction continue n'admet pas de dérivée symétrique*, Dissertationes Math. 130 (1976), 1–49.

Theorem

The family $\mathcal{C} \setminus \mathcal{SD}$ is strongly $\mathfrak{c}$-algebrable.



Summary

- $\mathcal{SC}$ – $2^{\mathfrak{c}}$-algebrable
- $\mathcal{SC} \setminus \mathcal{C}$ – $2^{\mathfrak{c}}$-algebrable
- $\mathcal{SC} \setminus \mathcal{SD}$ – $2^{\mathfrak{c}}$-algebrable
- $\mathcal{D}arb \setminus \mathcal{SD}$ – strong $2^{\mathfrak{c}}$-algebrable
- $\mathcal{D}arb \setminus \mathcal{SC}$ – strong $2^{\mathfrak{c}}$-algebrable
- $\mathcal{SD} \setminus \mathcal{D}arb$ – strong $\mathfrak{c}$-algebrable
- $\mathcal{SD} \setminus \mathcal{D}$ – strong $\mathfrak{c}$-algebrable
- $\mathcal{C} \setminus \mathcal{SD}$ – strong $\mathfrak{c}$-algebrable

Symmetric properties

L. Larson, *Symmetric real analysis: a survey*, Real Anal. Exchange 9 (1983/4), 154–178.

B.S. Thomson, *Symmetric Properties of Real Functions*, Monogr. Textbooks Pure Appl. Math. 183, Marcel Dekker, Inc., New York, 1994. xvi+447 pp.