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Introduction

In 2014 J. Borsík and J. Holos defined porouscontinuous functions. Using the notion of density in the sense of O'Malley in 2021 one introduced new definitions of porouscontinuity, namely $\mathcal{MO}_r$ and $\mathcal{SO}_r$-continuity. These kinds of porouscontinuity used upper porosity. We consider lower porouscontinuity in the sense of O'Malley, where lower porosity is used instead of standard (upper) porosity.

Notations

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural and the set of all real numbers, respectively.
By $\mathrm{cl}(A)$ we denote a closure of a set $A \subset \mathbb{R}$.
By $f_{\restriction A}$ we denote the restriction of $f\colon \mathbb{R} \to \mathbb{R}$ to $A \subset \mathbb{R}$.

Upper porosity

For a set $A \subset \mathbb{R}$ and an interval $I \subset \mathbb{R}$ let $\Lambda(A, I)$ denote the length of the largest open subinterval of I having an empty intersection with A. Then according to [2, 8], the right (upper) porosity of the set A at $x \in \mathbb{R}$ is defined as

$$p^+(A, x) = \limsup_{h \to 0^+} \frac{\Lambda(A, (x, x+h))}{h},$$

the left (upper) porosity of the set A at x is defined as

$$p^-(A, x) = \limsup_{h \to 0^+} \frac{\Lambda(A, (x-h, x))}{h},$$

and the (upper) porosity of A at x is defined as

$$p(A, x) = \max\left\{p^-(A, x), p^+(A, x)\right\}.$$

π_r-density and μ_r-density

Definition ([2])

Let $r \in [0, 1)$. A point $x \in \mathbb{R}$ will be called a point of π_r-density of a set $A \subset \mathbb{R}$ if $p(\mathbb{R} \setminus A, x) > r$.

Definition ([2])

Let $r \in (0, 1]$. A point $x \in \mathbb{R}$ will be called a point of μ_r-density of a set $A \subset \mathbb{R}$ if $p(\mathbb{R} \setminus A, x) \geq r$.

$\mathcal{P}_r$-continuity and $\mathcal{S}_r$-continuity (porouscontinuous functions)

Definition ([2])

Let $r \in [0,1)$ and $x \in \mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ will be called

- *$\mathcal{P}_r$-continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f_{\restriction A}$ is continuous at x;*
- *$\mathcal{S}_r$-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of π_r-density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.*

Symbols $\mathcal{P}_r(f)$ and $\mathcal{S}_r(f)$ denote the set of all points at which $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{P}_r$-continuous and $\mathcal{S}_r$-continuous, respectively.

$\mathcal{M}_r$-continuity and $\mathcal{N}_r$-continuity (porouscontinuous functions)

Definition ([2])

Let $r \in (0,1]$ and $x \in \mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ will be called

- $\mathcal{M}_r$*-continuous at x if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f_{\restriction A}$ is continuous at x;*
- $\mathcal{N}_r$*-continuous at x if for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of μ_r-density of A and $f(A) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$.*

Symbols $\mathcal{M}_r(f)$ and $\mathcal{N}_r(f)$ denote the set of all points at which $f\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{M}_r$-continuous and $\mathcal{N}_r$-continuous, respectively.

Relationships

Let $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$.

- If f is $\mathcal{P}_r$-continuous at x then f is $\mathcal{S}_r$-continuous at x for every $r \in [0,1)$.
- If f is $\mathcal{M}_r$-continuous at x then f is $\mathcal{N}_r$-continuous at x for every $r \in (0,1]$.
- If f is $\mathcal{M}_t$-continuous at x then f is $\mathcal{P}_r$-continuous at x for every $0 \le r < t \le 1$.
- If f is $\mathcal{N}_t$-continuous at x then f is $\mathcal{S}_r$-continuous at x for every $0 \le r < t \le 1$.

In [2] the equality $\mathcal{M}_r(f) = \mathcal{N}_r(f)$ for every f and every $r \in (0,1]$ was proved. Observe that if f is right-hand continuous or left-hand continuous at some x then f is porouscontinuous at x.

In [7] R. J. O'Malley modified the notion of preponderant continuity. Preponderant continuity is an another type of generalized continuity similar to porouscontinuity, in which a lower density of a Lebesgue measurable set at a point is used instead of porosity. R. J. O'Malley showed that one can replace density of a set by another condition involving the Lebesgue measure. Combining the notion of porouscontinuity defined by J. Borsík and J. Holos and using the concept of R. J. O'Malley we obtained another types of porouscontinuity.

πO_r-density and μO_r-density

Definition ([4])

Let $r \in [0,1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. A point x will be called a point of πO_r-density of a set A if for each $\eta > 0$ there exist $\delta \in (0,\eta)$ and an open interval $(a,b) \subset A \cap ((x-\delta, x+\delta) \setminus \{x\})$ such that $\frac{b-a}{\delta} > r$.

Definition ([4])

Let $r \in (0,1]$, $x \in \mathbb{R}$, $A \subset \mathbb{R}$. A point x will be called a point of μO_r-density of a set A if for each $\eta > 0$ there exist $\delta \in (0,\eta)$ and $(a,b) \subset A \cap ((x-\delta, x+\delta) \setminus \{x\})$ such that $\frac{b-a}{\delta} \geq r$.

Relationships

Remark ([4])

Let $r \in (0,1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of πO_r-density of A then x is a point of μO_r-density of A.

Remark ([4])

Let $r \in [0,1)$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of π_r-density of A then x is a point of πO_r-density of A.

Remark ([4])

Let $r \in (0,1]$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. If x is a point of μO_r-density of A then x is a point of μ_r-density of A.

$\mathcal{SO}_r$-continuity and $\mathcal{MO}_r$-continuity

Definition ([4])

Let $r \in [0,1)$, $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. We will say that f is $\mathcal{SO}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of πO_r-density of a set $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$.

Definition ([4])

Let $r \in (0,1]$, $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. We will say that f is $\mathcal{MO}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of μO_r-density of a set $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$.

Symbols $\mathcal{SO}_r(f)$ and $\mathcal{MO}_r(f)$ denote the set of all points at which f is $\mathcal{SO}_r$-continuous, $\mathcal{MO}_r$-continuous, respectively for corresponding r.

Notations

$\mathcal{Q}(f)$ and $\mathcal{C}(f)$ denote the set of all points at which f is quasi-continuous and continuous, respectively.
We introduce the following notations:

- $\mathcal{C} = \{f : \mathcal{C}(f) = \mathbb{R}\}$, $\mathcal{Q} = \{f : \mathcal{Q}(f) = \mathbb{R}\}$ and $\mathcal{C}^{\pm}$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that at every $x \in \mathbb{R}$, f is right-hand continuous or left-hand continuous (obviously $\mathcal{C} \subsetneq \mathcal{C}^{\pm}$),
- for $r \in (0, 1]$ let $\mathcal{M}_r = \{f : \mathcal{M}_r(f) = \mathbb{R}\}$, $\mathcal{MO}_r = \{f : \mathcal{MO}_r(f) = \mathbb{R}\}$,
- for $r \in [0, 1)$ let $\mathcal{P}_r = \{f : \mathcal{P}_r(f) = \mathbb{R}\}$, $\mathcal{S}_r = \{f : \mathcal{S}_r(f) = \mathbb{R}\}$ and $\mathcal{SO}_r = \{f : \mathcal{SO}_r(f) = \mathbb{R}\}$.

Main theorem

Theorem ([4])

Let $0 < r < t < 1$. *Then*

$$\mathcal{C}^{\pm} = \mathcal{MO}_1 \subset \mathcal{M}_1 \subset \mathcal{P}_t \subset \mathcal{S}_t \subset \mathcal{SO}_t \subset \mathcal{MO}_t \subset \mathcal{M}_t \subset \mathcal{P}_r \subset \\ \subset \mathcal{S}_r \subset \mathcal{SO}_r \subset \mathcal{MO}_r \subset \mathcal{M}_r \subset \mathcal{P}_0 \subset \mathcal{S}_0 \subset \mathcal{SO}_0 = \mathcal{Q}$$

and all inclusions are proper.

Lower porosity

Replacing upper limit by lower limit in the definition of porosity we obtain the right lower porosity of the set $A \subset \mathbb{R}$ at $x \in \mathbb{R}$ as

$$\underline{p}^{+}(A, x) = \liminf_{h \to 0^{+}} \frac{\Lambda(A, (x, x+h))}{h},$$

the left lower porosity as

$$\underline{p}^{-}(A, x) = \liminf_{h \to 0^{+}} \frac{\Lambda(A, (x-h, x))}{h},$$

and the lower porosity as

$$\underline{p}(A, x) = \min\left\{\underline{p}^{-}(A, x), \underline{p}^{+}(A, x)\right\}.$$

Example

Clearly, $\underline{p}^+(A,x) \leq p^+(A,x)$ and $\underline{p}^-(A,x) \leq p^-(A,x)$ for every $A \subset \mathbb{R}$ and $x \in \mathbb{R}$.

Example

There exists a set $A \subset \mathbb{R}$ such that $\underline{p}^+(A,0) = 0$ and $p^+(A,0) = 1$.

For example $A = \{0\} \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{(2n+1)!}, \frac{1}{(2n)!} \right)$.

Properties of lower porosity

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Theorem

Let $A \subset \mathbb{R}$ *and* $x \in \mathbb{R}$. *If there exists a decreasing sequence* $(x_n)_{n \in \mathbb{N}}$ *with terms belonging to* A *converging to* x *then for every* $\varepsilon > 0$ *there exists* $h \in (0, \varepsilon)$ *such that* $\frac{\Lambda(A,(x,x+h))}{h} < \frac{1}{2}$.

Corrolary

Let $A \subset \mathbb{R}$ *and* $x \in \mathbb{R}$. *If there exists a decreasing sequence* $(x_n)_{n \in \mathbb{N}}$ *with terms belonging to* A *converging to* x *then* $\underline{p}^+(A, x) \leq \frac{1}{2}$.

Properties of lower porosity

Similarly we can show that if there exists an increasing sequence $(x_n)_{n\in\mathbb{N}}$ with terms belonging to A converging to x then $\underline{p}^-(A,x) \leq \frac{1}{2}$.

Corrolary

For every $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ we have

$$\underline{p}(A,x) \leq \frac{1}{2} \text{ if } x \in \mathrm{cl}(A) \quad \text{and} \quad \underline{p}(A,x) = 1 \text{ if } x \notin \mathrm{cl}(A).$$

Properties of lower porosity

Lemma

For every $[a, b)$ *there exist two sequences* $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ *such that* $a < \ldots < b_{n+1} < a_n < b_n < \ldots < b$, $\lim_{n\to\infty} a_n = a$ *and*

$$\underline{p}^+\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty}(a_n, b_n), a\right) = \frac{1}{2}.$$

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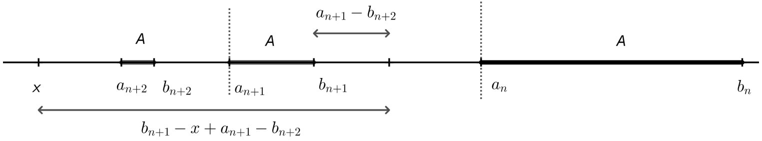
Lemma

Let $x < \ldots < b_{n+1} < a_n < b_n < \ldots < a_1 < b_1$ *be such that* $a_{n+1} - b_{n+2} < a_n - b_{n+1}$ *for every* $n \geq 1$, $\lim_{n\to\infty} a_n = a$ *and let* $A = \bigcup_{n=1}^{\infty}[a_n, b_n]$. *Then*

$$\inf_{h\in[a_{n+1}-x, a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} = \frac{a_{n+1}-b_{n+2}}{b_{n+1}-x+a_{n+1}-b_{n+2}}.$$



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Corrolary

Let $A = \bigcup_{n=1}^{\infty}[a_n, b_n]$, *where* $x < \ldots < b_{n+1} < a_n < b_n < \ldots < a_1 < b_1$, $\lim_{n\to\infty} a_n = x$ *and* $a_{n+1} - b_{n+2} < a_n - b_{n+1}$ *for every* $n \geq 1$. *Then*

$$\underline{p}^{+}(A, x) = \liminf_{n\to\infty} \frac{a_{n+1} - b_{n+2}}{b_{n+1} - x + a_{n+1} - b_{n+2}}.$$

Properties of lower porosity

Theorem

For every $x \in \mathbb{R}$ *and for every sequence* $(c_n)_{n\geq 1}$ *from* $(0, \frac{1}{2})$ *there exist two sequences* $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$ *such that* $\lim_{n\to\infty} a_n = x$, $x < \ldots < b_{n+1} < a_n < b_n < \ldots < a_1 < b_1$, $a_{n+1} - b_{n+2} < a_n - b_{n+1}$ *for every* $n \geq 1$ *and* $\inf_{h\in[a_{n+1}-x, a_n-x]} \frac{\Lambda(A,(x,x+h))}{h} = c_n$, *where* $A = \bigcup_{n=1}^{\infty}[a_n, b_n]$.

Corrolary

For every $c, d \in [0, \frac{1}{2}]$ *and* $x \in \mathbb{R}$ *there exists* $A \subset \mathbb{R}$ *such that* $\underline{p}^{+}(A, x) = c$ *and* $\underline{p}^{-}(A, x) = d$.

$\underline{\pi}_r$-density and $\underline{\mu}_r$-density

Applying lower porosity and O'Malley concept of preponderant continuity we can define another types of porouscontinuity. We define these types of porouscontinuity only for $r \in [0, \frac{1}{2}] \cup \{1\}$.

Definition

Let $r \in [0, \frac{1}{2}]$. A point $x \in \mathbb{R}$ will be called a point of $\underline{\pi}_r$-density of a set $A \subset \mathbb{R}$ if $\underline{p}(\mathbb{R} \setminus A, x) > r$.

Definition

Let $r \in (0, \frac{1}{2}] \cup \{1\}$. A point $x \in \mathbb{R}$ will be called a point of $\underline{\mu}_r$-density of a set $A \subset \mathbb{R}$ if $\underline{p}(\mathbb{R} \setminus A, x) \geq r$.

$\underline{\pi O}_r$-density and $\underline{\mu O}_r$-density

Definition

Let $r \in [0, \frac{1}{2}]$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. A point x will be called a point of $\underline{\pi O}_r$-density of a set A if there exists $\eta > 0$ such that for every $\delta \in (0, \eta)$ we can find open intervals $(a_1, b_1) \subset A \cap (x - \delta, x)$ and $(a_2, b_2) \subset A \cap (x, x + \delta)$ for which $\frac{b_1 - a_1}{\delta} > r$ and $\frac{b_2 - a_2}{\delta} > r$.

Definition

Let $r \in (0, \frac{1}{2}] \cup \{1\}$, $x \in \mathbb{R}$, $A \subset \mathbb{R}$. A point x will be called a point of $\underline{\mu O}_r$-density of a set A if there exists $\eta > 0$ such that for every $\delta \in (0, \eta)$ we can find open intervals $(a_1, b_1) \subset A \cap (x - \delta, x)$ and $(a_2, b_2) \subset A \cap (x, x + \delta)$ for which $\frac{b_1 - a_1}{\delta} \geq r$ and $\frac{b_2 - a_2}{\delta} \geq r$.

Relationships

Remark

Let $x \in \mathbb{R}$ and $A \subset \mathbb{R}$.

- If x is a point of $\underline{\pi}_r$-density of A then x is a point of $\underline{\pi O}_r$-density of A for $r \in [0, \frac{1}{2}]$.
- If x is a point of $\underline{\pi O}_r$-density of A then x is a point of $\underline{\mu O}_r$-density of A for $r \in (0, \frac{1}{2}]$.
- If x is a point of $\underline{\mu O}_r$-density of A then x is a point of $\underline{\mu}_r$-density of A for $r \in (0, \frac{1}{2}] \cup \{1\}$.

$\underline{\mathcal{SO}}_r$-continuity, $\underline{\mathcal{S}}_r$-continuity, $\underline{\mathcal{P}}_r$-continuity

Definition

Let $r \in [0, \frac{1}{2}]$, $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. We will say that f is

- $\underline{\mathcal{SO}}_r$*-continuous at x if for each $\varepsilon > 0$, the point x is a point of $\underline{\pi O}_r$-density of a set $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$;*
- $\underline{\mathcal{S}}_r$*-continuous at x if for each $\varepsilon > 0$, the point x is a point of $\underline{\pi}_r$-density of a set $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$;*
- $\underline{\mathcal{P}}_r$*-continuous at x if there exists $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $\underline{\pi}_r$-density of A and $f_{\restriction A}$ is continuous at x.*

Symbols $\underline{\mathcal{SO}}_r(f)$, $\underline{\mathcal{S}}_r(f)$ and $\underline{\mathcal{P}}_r(f)$ denote the set of all points at which f is $\underline{\mathcal{SO}}_r$-continuous, $\underline{\mathcal{S}}_r$-continuous and $\underline{\mathcal{P}}_r$-continuous, respectively.

$\underline{\mathcal{MO}}_r$-continuity, $\underline{\mathcal{M}}_r$-continuity, $\underline{\mathcal{N}}_r$-continuity

Definition

Let $r \in (0, \frac{1}{2}] \cup \{1\}$, $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. We will say that f is

- *$\underline{\mathcal{MO}}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of $\underline{\mu O}_r$-density of a set $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$;*
- *$\underline{\mathcal{M}}_r$-continuous at x if for each $\varepsilon > 0$, the point x is a point of $\underline{\mu}_r$-density of a set $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$;*
- *$\underline{\mathcal{N}}_r$-continuous at x if there exists $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $\underline{\mu}_r$-density of A and $f_{\restriction A}$ is continuous at x.*

Symbols $\underline{\mathcal{MO}}_r(f)$, $\underline{\mathcal{M}}_r(f)$ and $\underline{\mathcal{N}}_r(f)$ denote the set of all points at which f is $\underline{\mathcal{MO}}_r$-continuous, $\underline{\mathcal{M}}_r$-continuous and $\underline{\mathcal{N}}_r$-continuous, respectively.

Notations

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- For $r \in [0, \frac{1}{2}]$ let $\underline{\mathcal{SO}}_r = \{f : \underline{\mathcal{SO}}_r(f) = \mathbb{R}\}$, $\underline{\mathcal{S}}_r = \{f : \underline{\mathcal{S}}_r(f) = \mathbb{R}\}$ and $\underline{\mathcal{P}}_r = \{f : \underline{\mathcal{P}}_r(f) = \mathbb{R}\}$.
- For $r \in (0, \frac{1}{2}] \cup \{1\}$ let $\underline{\mathcal{MO}}_r = \{f : \underline{\mathcal{MO}}_r(f) = \mathbb{R}\}$, $\underline{\mathcal{M}}_r = \{f : \underline{\mathcal{M}}_r(f) = \mathbb{R}\}$ and $\underline{\mathcal{N}}_r = \{f : \underline{\mathcal{N}}_r(f) = \mathbb{R}\}$.

Relationships

Proposition

For every $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ the following properties hold:

- f is $\underline{\mathcal{N}}_r$-continuous at x if and only if f is $\underline{\mathcal{M}}_r$-continuous at x for every $r \in (0, \frac{1}{2}] \cup \{1\}$.
- If f is $\underline{\mathcal{S}}_r$-continuous at x then f is $\underline{\mathcal{SO}}_r$-continuous at x for every $r \in [0, \frac{1}{2})$.
- If f is $\underline{\mathcal{SO}}_r$-continuous at x then f is $\underline{\mathcal{MO}}_r$-continuous at x for every $r \in (0, \frac{1}{2})$.

Relationships

Proposition

For every $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ the following properties hold:

- If f is $\underline{\mathcal{MO}}_r$-continuous at x then f is $\underline{\mathcal{M}}_r$-continuous at x for every $r \in (0, \frac{1}{2}]$.
- If f is $\underline{\mathcal{P}}_r$-continuous at x then f is $\underline{\mathcal{S}}_r$-continuous at x for every $r \in [0, \frac{1}{2})$.
- If f is $\underline{\mathcal{M}}_t$-continuous at x then f is $\underline{\mathcal{P}}_r$-continuous at x for every $0 \leq r < t \leq \frac{1}{2}$.

Relationships

Theorem

Let $r \in [0, \frac{1}{2})$, $x \in \mathbb{R}$ *and* $f \colon \mathbb{R} \to \mathbb{R}$. *Then* f *is* $\underline{\mathcal{P}}_r$*-continuous at* x *if and only if* $\lim_{\varepsilon \to 0^+} \underline{p}(\mathbb{R} \setminus \{y \in \mathbb{R} \colon |f(x) - f(y)| < \varepsilon\}, x) > r$.

Theorem

Let $r \in [0, \frac{1}{2}]$, $x \in \mathbb{R}$. *A function* $f \colon \mathbb{R} \to \mathbb{R}$ *is* $\underline{\mathcal{SO}}_r$*-continuous at* x *if and only if there exists a set* $A \subset \mathbb{R}$ *such that* $x \in A$, x *is a point of* $\underline{\pi O}_r$*-density of* A *and* $f_{\restriction A}$ *is continuous at* x.

Relationships

Theorem

Let $r \in (0, \frac{1}{2}] \cup \{1\}$, $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is $\underline{\mathcal{MO}}_r$-continuous at x if and only if there exists a set $A \subset \mathbb{R}$ such that $x \in A$, x is a point of $\underline{\mu O}_r$-density of A and $f_{\restriction A}$ is continuous at x.

Relationships

Proposition

For every $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ the following conditions are equivalent:

- f is $\underline{\mathcal{M}}_1$-continuous at x;
- f is $\underline{\mathcal{MO}}_{\frac{1}{2}}$-continuous at x;
- f is $\underline{\mathcal{SO}}_{\frac{1}{2}}$-continuous at x;
- f is $\underline{\mathcal{S}}_{\frac{1}{2}}$-continuous at x;
- f is $\underline{\mathcal{P}}_{\frac{1}{2}}$-continuous at x;
- f is continuous at x.

Relationships

Proposition

A function $f\colon \mathbb{R} \to \mathbb{R}$ is $\underline{\mathcal{SO}}_0$-continuous at $x \in \mathbb{R}$ if and only if f is bilateral quasi-continuous at x. Then $\mathcal{Q} \supset \mathcal{Q}^{bil} = \underline{\mathcal{SO}}_0$, where $\mathcal{Q}^{bil}$ denotes the family of functions which are bilateral quasi-continuous at every point.
($f\colon \mathbb{R} \to \mathbb{R}$ is bilateral quasi-continuous at x if for every $\varepsilon > 0$ and $\delta > 0$ there exist $(a,b) \subset (x-\delta, x)$ and $(c,d) \subset (x, x+\delta)$ such that $f((a,b) \cup (c,d)) \subset (f(x)-\varepsilon, f(x)+\varepsilon)$.)

Relationships

Theorem

Let $r \in (0, \frac{1}{2}]$. *Then* $\underline{\mathcal{P}}_r \subset \underline{\mathcal{S}}_r \subset \underline{\mathcal{SO}}_r \subset \underline{\mathcal{MO}}_r \subset \underline{\mathcal{M}}_r$.

Relationships

Theorem

$\mathcal{C} = \underline{\mathcal{M}}_1 = \underline{\mathcal{MO}}_{\frac{1}{2}} = \underline{\mathcal{S}}_{\frac{1}{2}} = \underline{\mathcal{SO}}_{\frac{1}{2}} = \underline{\mathcal{P}}_{\frac{1}{2}}$ *and* $\underline{\mathcal{S}}_0 \subset \underline{\mathcal{SO}}_0 = \mathcal{Q}^{bil}$.

Main theorem

Theorem

Let $0 < r < t < \frac{1}{2}$. *Then*

$$\mathcal{C} = \underline{\mathcal{M}}_1 = \underline{\mathcal{P}}_{\frac{1}{2}} = \underline{\mathcal{S}}_{\frac{1}{2}} = \underline{\mathcal{SO}}_{\frac{1}{2}} = \underline{\mathcal{MO}}_{\frac{1}{2}} \subset \underline{\mathcal{M}}_{\frac{1}{2}} \subset \underline{\mathcal{P}}_t \subset \underline{\mathcal{S}}_t \subset$$
$$\subset \underline{\mathcal{SO}}_t \subset \underline{\mathcal{MO}}_t \subset \underline{\mathcal{M}}_t \subset \underline{\mathcal{P}}_r \subset \underline{\mathcal{S}}_r \subset \underline{\mathcal{SO}}_r \subset \underline{\mathcal{MO}}_r \subset$$
$$\subset \underline{\mathcal{M}}_r \subset \underline{\mathcal{P}}_0 \subset \underline{\mathcal{S}}_0 \subset \underline{\mathcal{SO}}_0 = \mathcal{Q}^{bil}$$

and all inclusions are proper.

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Thank you for your attention!