Universal families and the convergence in category.

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17 - 21 June 2024

Theorem (F.Riesz) If a sequence $\{f_n\}_{n\in N}$ of real-valued measurable functions defined on $[0,1]$ converges in measure to a function f, then for each subsequence $\left\{f_{n_m}\right\}_{m\in N}$ there exists a subsequence $\left\{f_{n_{m_p}}\right\}_{p\in N}$ which converges a.e. to f.

Definition: We say that the sequence $\{f_n\}_{n\in N}$ of functions having the Baire property converges in category to a function f (having the Baire property) if and only if for each subsequence $\left\{f_{n_m}\right\}_{n\in N}$ there exists a subsequence $\left\{f_{n_{m_p}}\right\}_{p\in N}$ convergent to f except on a set of the first category.

Convergence in measure is metrizable:

$$d(f,g)=\int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}dx$$

Convergence in category does not generate a Fréchet topology.

Theorem (D.Egorov) If $\{f_n\}_{n\in N}$ converges to f a.e., then for each $\varepsilon>0$ there exists $E_\varepsilon\subset[0,1]$ such that $\lambda([0,1]\setminus E_\varepsilon)<\varepsilon$ and $f_n|E_\varepsilon \rightrightarrows f|E_\varepsilon$.

Theorem: There exists a sequence $\{f_n\}_{n\in N}$ of functions having the Baire property such that $f_n\to f$ except on a set of the first category and if $E\subset[0,1]$ is such that $f_n|E\rightrightarrows f|E$, then E is of the first category.

$([0,1], S, J)$ - a measurable space equipped with a σ-ideal $J \subset S$.

Definition: A sequence $\{f_n\}_{n \in N}$ is called a basic sequence if for each S-measurable function f there exists a sequence of finite linear combinations of functions from $\{f_n\}_{n \in N}$ which converges to f except on a set from J.

$\mathcal{M}$ - the space of equivalence classes of measurable functions (defined on $[0,1]$).

Theorem: The dual space $\mathcal{M}'$ of $\mathcal{M}$ consists only of 0.

Theorem: If $\{f_n\}_{n \in N}$ is a basic sequence in $\mathcal{M}$, it remains basic if any finite subset is deleted.

C.Goffman, D.Waterman, Basic sequences in the space of measurable functions, PAMS 11 (1960), 211-213.

Definition: If $\{f_n\}_{n\in N}$ is a basic sequence in the space $\mathcal{M}$ then a series $\sum_{n=1}^{\infty} a_n f_n$ is called universal in $\mathcal{M}$ if for each $f \in \mathcal{M}$ there exists a subsequence of partial sums of this series which converges to f a.e.

Theorem (Goffman, Waterman) There exists the universal series in $\mathcal{M}$.

The proof depends on the above mentioned distance function.

Theorem: There exists the universal series in the space of functions having the Baire property $\left(\text{defined on } \left[\frac{1}{2}, 1\right]\right)$.

The proof uses the fact that if f has the Baire property, then there exists a sequence $\{f_n\}_{n\in N}$ of continuous functions convergent to f except on a set of the first category.

Theorem (J.Pal, W.Sierpiński) If $f : \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ is continuous, then for each $\varepsilon > 0$ and $m \in N$ there exists a polynomial p such that $a_0 = a_1 = \ldots = a_m = 0$ such that $|f(x) - p(x)| < \varepsilon$ for each $x \in \left[\frac{1}{2}, 1\right]$.

J.Pal, Über eine Anwendung des Weierstrass schen Satzes von der Annäherung stetiger Funktionen durch Polynome, Tohoku Math. J. 5 (1914), 8-9.

W.Sierpiński Sur une serie universelle pour les fonctions continues, Studia Math. 7 (1937), 45-48 or OC. t. I, 292-295.

Theorem: There exists a sequence $\{a_n\}_{n\in N}$ of rational numbers such that for each continuous function f with $f(0) = 0$ there exists an increasing sequence $\{n_k\}_{k\in N}(n_0 = 0)$ such that $f(x) = \sum_{k=0}^{\infty}\left(\sum_{n=n_k+1}^{n_{k+1}} a_n x^n\right)$ (uniform convergence).

More informations:

Karl-Goswin Grosse-Erdmann, Universal Families and Hypercyclic operators, Bull. Amer. Math. Soc. 36(3) (1999), 345-381.

Thank you very much