Dominated Sets

Hausdorff Measures

Microscopic Sets

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MICROSCOPIC SETS – HISTORY

DEF (Appell 2001) A set $X \subseteq \mathbb{R}$ is microscopic if
$\forall \varepsilon > 0 \; \exists \{E_n : n \in \omega_+\}$ → covers X
→ $\operatorname{diam} E_n \leq \varepsilon^n$

→ Appell, D'Aniello, Väth 2001
→ micro($\mathbb{R}$) is a σ-ideal
→ micro($\mathbb{R}$) $\subsetneq$ HDZ (Hausdorff dim zero)

→ Horbaczewska, Karasińska, Wagner-Bojakowska 2013
→ micro is G_δ-based
→ $X \subseteq \mathbb{R}$ is micro $\iff \forall \varepsilon > 0 \; \exists \{E_n : n \in \omega^+\}$ λ-cover
→ Smz $\subsetneq$ micro $\subsetneq$ HDZ
→ $\mathbb{R} = F \cup M$
meager microscopic
→ $\exists M \subseteq \mathbb{R}$ microscopic $M + M = \mathbb{R}$

MICROSCOPIC SETS – HISTORY

- Horbaczewska 2014
  - generalization: $\operatorname{diam} E_n \leq f_n(\varepsilon)$ (in place of ε^n)

    (sequence $\langle f_n \rangle$ satisfying some properties)
  - $f_n(\varepsilon) = \varepsilon^{2^n}$: "closed f_n-microscopic" sets form a σ-ideal
- Czudek, Kwela, Mrożek, Wołoszyn 2016
  - nanoscopic sets: ε^{2^n}
  - picoscopic sets: $\varepsilon^{n!}$

  NOT ideals!
- Głąb, Chrząszcz 2015
  - estimates of cardinal invariants of $\mathrm{micro}(\mathbb{R})$
- Kwela 2016
  - $\mathrm{add}\ \mathrm{micro}(\mathbb{R}) = \omega_1$
  - $\mathcal{M}_{\ln}$ .... $\varepsilon^{\ln(n+1)}$

MICROSCOPIC SETS – HISTORY

→ Karasińska, Paszkiewicz, Wagner-Bojakowska 2017

DEF Let $s = \langle s_n : n \in \omega_+ \rangle \in \ell^1$, $s_n \searrow 0$:

$X \subseteq \mathbb{R}$ is s-microscopic if

$\forall k\ \exists \{E_n : n \in \omega_+\}$ → cover of X

→ $\operatorname{diam} E_n \leq s_{kn}$

→ s-micro is a σ-ideal, G_δ-based

→ $\mathcal{S}mz \subsetneq s\text{-micro} \subsetneq \mathcal{N}$

→ $\mathrm{Leb}(N) = 0 \Rightarrow \exists s$ N is s-micro

→ Horbaczewska 2018

DEF Let $\mathcal{S} \subseteq \ell^1$ be a set of sequences decreasing to zero.

$X \subseteq \mathbb{R}$ is $\mathcal{S}$-microscopic if $\forall s \in \mathcal{S}\ \exists \{E_n\}$ ....

→ generalizes both previous generalizations

→ Paszkiewicz Conjecture

$\forall p \in [0,1]\ \exists s \ \cdots\ \{X \subseteq \mathbb{R} : \dim X \leq p\} = s\text{-micro}$

DOMINATED SETS

- $\mathcal{S}$ ... all sequences strictly decreasing to zero
- $s \in \mathcal{S}$ s-fine cover: $\operatorname{diam} E_n < s_n$
- $S \subseteq \mathcal{S}$ S-dominated set: $\forall s \in S\ \exists$ s-fine cover
  - $\mathcal{D}(S)$ ... all S-dominated sets in a given space X
  - $\mathcal{E}(S) = \{A \subseteq X : \exists B \in \mathcal{D}(S),\ B\ \sigma\text{-cpt},\ B \supseteq A\}$

DOMINATED SETS

→ $\mathbb{S}$ ... all sequences strictly decreasing to zero

→ $s \in \mathbb{S}$ s-fine cover: $\operatorname{diam} E_n < s_n$

→ $\mathcal{S} \subseteq \mathbb{S}$ $\mathcal{S}$-dominated set: $\forall s \in \mathcal{S}\ \exists$ s-fine cover

→ $\mathcal{D}(\mathcal{S})$ ... all $\mathcal{S}$-dominated sets in a given space X

→ $\mathcal{E}(\mathcal{S}) = \{A \subseteq X : \exists B \in \mathcal{D}(\mathcal{S}),\ B\ \sigma\text{-cpt},\ B \supseteq A\}$

→ $s \in \mathbb{S}$ ..., $s^{(k)}_n = s_{k \cdot n}$ $s^{+k}_n = s_{n+k}$

→ $\mathcal{S} \subseteq \mathbb{S}$ is closed: $\forall s \in \mathcal{S}\ \exists s' \in \mathcal{S}\ \ s' \leq s^{(2)}$

shift-complete: .... $s' \leq s^{+1}$

→ closure: $\overline{\mathcal{S}} = \{s^{(k)} : s \in \mathcal{S},\ k \in \omega_+\}$

shift-completion: $\mathcal{S}^+ = \{s^{+k},\ s \in \mathcal{S},\ k \in \omega\}$

THM → $\mathcal{S}$ closed $\Rightarrow \mathcal{D}(\mathcal{S})$ is a σ-ideal

→ $\mathcal{S}$ shift-complete $\Rightarrow \mathcal{E}(\mathcal{S})$ is a σ-ideal

DOMINATED SETS

- $\mathbb{S}$ ... all sequences strictly decreasing to zero
- $s \in \mathbb{S}$ s-fine cover: $\operatorname{diam} E_n < s_n$
- $S \subseteq \mathbb{S}$ S-dominated set: $\forall s \in S\ \exists$ s-fine cover
  - $\mathcal{D}(S)$ ... all S-dominated sets in a given space X
  - $\mathcal{E}(S) = \{A \subseteq X : \exists B \in \mathcal{D}(S),\ B\ \sigma\text{-cpt},\ B \supseteq A\}$
- $s \in \mathbb{S}$ ..., $s_n^{(k)} = s_{k \cdot n}$ $s_n^{+k} = s_{n+k}$
- $S \subseteq \mathbb{S}$ is closed: $\forall s \in S\ \exists s' \in S\ \ s' \le s^{(2)}$

  shift-complete: .... $s' \le s^{+1}$
- closure: $\overline{S} = \{s^{(k)} : s \in S,\ k \in \omega_+\}$

  shift-completion: $S^+ = \{s^{+k} : s \in S,\ k \in \omega\}$

THM
- S closed $\Rightarrow \mathcal{D}(S)$ is a σ-ideal
- S shift-complete $\Rightarrow \mathcal{E}(S)$ is a σ-ideal

EX (Horbaczewska 2014, Czudek et al. 2016)

$S = \{\langle \varepsilon 2^n \rangle : \varepsilon > 0\}$
- $\mathcal{E}(S)$ is σ-ideal
- $\mathcal{D}(S)$ is not an ideal

DOMINATED SETS

PROP If $\mathcal{S} \subseteq \$$ is closed, then
E is $\mathcal{S}$-dominated $\Leftrightarrow \forall s \in \mathcal{S}\ \exists$ s-fine λ-cover
$\forall x\, \exists^\infty n\ x \in E_n$

EX's $\rightarrow$ let $g = \langle 2^{-n} : n \in \omega_+ \rangle$: $\mathcal{D}(\underline{g}) = \text{micro}(X)$
geometric sequences

$\rightarrow$ let $h^p = \langle \frac{1}{(n+1)^p} : n \in \omega_+ \rangle$, $p > 0$. harmonic seq's
$\mathcal{S} = \{ h^p, p > 0 \}$

$\mathcal{D}(\mathcal{S}) = \mathcal{D}(\{ \varepsilon^{\ln(n+1)} : \varepsilon > 0 \}) = \mathcal{M}_{\ln}$ (Kwela)

$\rightarrow$ $\mathcal{D}(\$) = \text{Smz}$
$\mathcal{E}(\$) =$ countable sets
all sequences

SHARPLY DOMINATED SETS

DEF Let $I \subseteq \omega_+$. Then $\langle E_n : n \in I \rangle$ is <u>s-fine</u> if $\forall n \in I$ $\operatorname{diam} E_n < s_n$.

→ Summable ideal: $\mathcal{I}_{1/n} = \{ I \subseteq \omega_+ : \sum_{n \in I} \frac{1}{n} < \infty \}$

DEF Let $\mathcal{S} \subseteq \mathbb{S}$ be closed. A set E is <u>sharply $\mathcal{S}$-dominated</u> if $\forall s \in \mathcal{S}$ $\exists I \in \mathcal{I}_{1/n}$ and an s-fine λ-cover $\langle E_n : n \in I \rangle$.

Notation: $\mathcal{D}^*(\mathcal{S})$

THM Let $\mathcal{S} \subseteq \mathbb{S}$ be closed. Then $\mathcal{D}^*(\mathcal{S})$ is a σ-ideal and

$$\begin{tikzcd} & \mathcal{D}(\mathcal{S}) & \\ \mathcal{D}^*(\mathcal{S}) \arrow[ur] & & \mathcal{E}(\mathcal{S}) \arrow[ul] \end{tikzcd}$$

and no other inclusions hold for all closed $\mathcal{S}$.

HAUSDORFF MEASURES

→ Gauges: $\phi \prec \psi \overset{\text{def}}{\equiv} \lim\limits_{r \to 0} \frac{\psi(r)}{\phi(r)} = 0$

→ $\phi \preceq \psi$

→ $\phi \approx \psi$

→ $s \asymp \phi \overset{\text{def}}{\equiv} \exists a, b > 0 \ \forall n \quad a \leq n\phi(s_n) \leq b.$ $\qquad \phi(s_n) \cong \frac{1}{n}$

→ $\phi \asymp s \asymp \psi \Rightarrow \phi \approx \psi$

$s \asymp \phi \approx \psi \asymp t \Rightarrow \mathcal{D}(\bar{s}) = \mathcal{D}(\bar{t}), \ \mathcal{D}^{\#}(\bar{s}) = \mathcal{D}^{\#}(\bar{t}) \ldots$

Ex's → $\tau(r) = \frac{1}{\log \frac{1}{r}}$ .... $\tau \asymp \overset{\mapsto 2^{-n}}{g}$ .... microscopic sets

→ $\phi(r) = r^p$ .... $\phi \asymp \left\langle \frac{1}{(n+1)^{1/p}} \right\rangle$

HAUSDORFF MEASURES

- Gauges: $\phi \prec \psi \overset{\text{def}}{\equiv} \lim_{r\to 0} \frac{\psi(r)}{\phi(r)} = 0$
  - $\phi \preceq \psi$
  - $\phi \approx \psi$
- $s \asymp \phi \overset{\text{def}}{\equiv} \exists a,b>0 \ \forall n \quad a \leq n\phi(s_n) \leq b.$ $\quad \phi(s_n) \cong \frac{1}{n}$
- $\phi \asymp s \asymp \psi \Rightarrow \phi \approx \psi$

  $s \asymp \phi \approx \psi \asymp t \Rightarrow \mathcal{D}(\bar{s}) = \mathcal{D}(\bar{t})\,,\ \mathcal{D}^\#(\bar{s}) = \mathcal{D}^\#(\bar{t}) \ \ldots$

Ex's
- $\tau(r) = \frac{1}{\log \frac{1}{r}}$ .... $\tau \asymp g$ .... microscopic sets $\quad (\to 2^{-n})$
- $\phi(r) = r^p$ .... $\phi \asymp \langle \frac{1}{(n+1)^{1/p}} \rangle$

THM $\forall\ \phi \asymp s$, then $\mathcal{D}^\#(\bar{s}) = \mathcal{N}(\mathcal{H}^\phi) = \{E : \mathcal{H}^\phi(E) = 0\}.$

Proof $\mathcal{H}^\phi(E) = 0 \Rightarrow \exists\ \lambda\text{-cover}\ \sum \phi(\operatorname{diam} E_n) < 1 \overset{\text{WLOG}}{\Longrightarrow} \operatorname{diam} E_n$ decreases

$\Rightarrow \phi(\operatorname{diam} E_n) < \frac{1}{n} \Rightarrow \operatorname{diam} E_n < s_n$

HAUSDORFF MEASURES

THM Let $s \asymp \phi$.

- $\exists E \subseteq \mathbb{Z}^\omega$, $E \in \mathcal{E}(\bar{s}) \setminus \mathcal{D}^\#(\bar{s})$
- If $\phi < 1$, then $\exists E \subseteq \mathbb{R}$, $E \in \mathcal{E}(\bar{s}) \setminus \mathcal{N}(\mathcal{H}^\phi)$

Ex

$$\left.\begin{aligned} s_n &= \frac{1}{n \log n} \\ \phi(s_n) &= \frac{1}{n} \end{aligned}\right\} \quad s \asymp \phi$$

- $\phi < 1$ hence $\mathcal{H}^\phi([0,1])$ not σ-finite
- $\sum s_n^{(k)} = \sum \frac{1}{kn \log(kn)} = \infty$ for all k

hence $[0,1]$ is $\bar{s}$-dominated.

- $\mathcal{E}(\bar{s}) \not\subseteq \mathcal{N}_f(\mathcal{H}^\phi)$

↳ sets of σ-finite measure

HAUSDORFF MEASURES

THM $(X=\omega^\omega)$ $\forall s\in\mathfrak{S}$, $\phi\asymp s$ $\quad\mathcal{E}(\bar{s})\notin \mathcal{N}_\sigma(\mathcal{H}^\phi)$

$(X=\mathbb{R})$ $\forall s\in\mathfrak{S}$, $\phi\asymp s$ such that $\frac{r}{\phi(r)}$ decreases to zero

$$\mathcal{E}(\bar{s})\notin \mathcal{N}_\sigma(\mathcal{H}^\phi)$$

$\longrightarrow$ In both cases: $\mathcal{D}(\bar{s})\neq \mathcal{N}_\sigma(\mathcal{H}^\phi)$

Question: If $s\asymp\phi$: $\mathcal{N}_\sigma(\mathcal{H}^\phi)\subseteq\mathcal{D}(\bar{s})$?

HAUSDORFF DIMENSION

THM Let ϕ be a gauge, $s \in \mathcal{S}$. Suppose $\sum \phi(s_n) < \infty$.

→ $\mathcal{D}(s) \subseteq \mathcal{N}(\mathcal{H}^\phi)$

→ $(X = \mathbb{R})$ If $\mathcal{H}^\phi(\mathbb{R}) > 0$, then $\mathcal{D}(s) \subsetneq \mathcal{N}(\mathcal{H}^\phi)$

→ Particular case: $p > 0$, $\phi_p(r) = r^p$, $h = \langle \frac{1}{n+1} \rangle$

$p < q$: $\mathcal{N}(\mathcal{H}^p) = \mathcal{D}^\#(\overline{h^{1/p}}) \subsetneq \mathcal{D}(\overline{h^{1/p}}) \subsetneq \mathcal{N}(\mathcal{H}^q) \subsetneq \mathcal{D}(\overline{h^{1/q}})$

→ $\{E \subseteq X : \dim_H E \leq p\} = \bigcap_{q>p} \mathcal{N}(\mathcal{H}^q) = \bigcap_{q>p} \mathcal{D}(\overline{h^{1/q}})$

Let $\mathcal{S} = \{h^q : pq < 1\}$: $\{E : \dim_H E \leq p\} = \mathcal{D}(\overline{\mathcal{S}})$

HAUSDORFF DIMENSION

THM Let ϕ be a gauge, $s \in \mathcal{S}$. Suppose $\sum \phi(s_n) < \infty$.

$\to$ $\mathcal{D}(\bar{s}) \subseteq \mathcal{N}(\mathcal{H}^\phi)$

$\to$ $(X = \mathbb{R})$ If $\mathcal{H}^\phi(\mathbb{R}) > 0$, then $\mathcal{D}(\bar{s}) \subsetneq \mathcal{N}(\mathcal{H}^\phi)$

$\to$ Particular case: $p > 0$, $\phi_p(r) = r^p$, $h = \langle \frac{1}{n+1} \rangle$

$$p<q: \ \mathcal{N}(\mathcal{H}^p) = \mathcal{D}^{\#}(\overline{h^{1/p}}) \subsetneq \mathcal{D}(\overline{h^{1/p}}) \subsetneq \mathcal{N}(\mathcal{H}^q) \subsetneq \mathcal{D}(\overline{h^{1/q}})$$

$\to$ $\{E \subseteq X : \dim_H E \le p\} = \bigcap_{q>p} \mathcal{N}(\mathcal{H}^q) = \bigcap_{q>p} \mathcal{D}(\overline{h^{1/q}})$

Let $\mathcal{S} = \{h^q : pq < 1\}$: $\{E : \dim_H E \le p\} = \mathcal{D}(\bar{\mathcal{S}})$

$\to$ $\mathrm{HDZ} = \mathcal{D}(\underbrace{\{h^p : p > 0\}}_{\text{closed}}) = \mathcal{M}_{ln}$

$\to$ (Kwela) Estimate add $\mathcal{M}_{ln}$.

$$\operatorname{add} \mathcal{M}_{ln} = \operatorname{add} \mathrm{HDZ} \ge \min_{p>0} \operatorname{add} \mathcal{N}(\mathcal{H}^p) = \operatorname{add} \mathcal{N}$$

PASZKIEWICZ CONJECTURE

THM $\forall s \in \mathcal{S}$ $\quad HDZ \neq \mathcal{D}(\bar{s})$

LEMMA $(X = \mathbb{R})$ $\quad \mathcal{N}(\mathcal{H}^{\phi}) \subseteq HDZ \Rightarrow \mathcal{N}(\mathcal{H}^{\phi^2}) \subseteq HDZ$

Proof Suppose $HDZ = \mathcal{D}(\bar{s})$, $\phi \asymp s$. Then

$$\mathcal{N}(\mathcal{H}^{\phi}) \subseteq \mathcal{D}(\bar{s}) = HDZ$$

Since $\sum \phi^2(s_n) = \sum \frac{1}{n^2} < \infty$; $\quad \mathcal{D}(\bar{s}) \subseteq \mathcal{N}(\mathcal{H}^{\phi^2}) \subseteq HDZ$.

Hence:

$$\mathcal{D}(\bar{s}) \subseteq \underbrace{\mathcal{N}(\mathcal{H}^{\phi^2}) \subseteq \mathcal{N}(\mathcal{H}^{\phi^3}) \subseteq \mathcal{N}(\mathcal{H}^{\phi^4})}_{\text{all equal}} \subseteq HDZ,$$

Howroyd: $\exists E \subseteq \mathbb{R}$ $\qquad 0 < \mathcal{H}^{\phi^3}(E) < \infty$

$$\left.\begin{aligned} \Rightarrow \quad & \mathcal{H}^{\phi^2}(E) = \infty \\ & \mathcal{H}^{\phi^4}(E) = 0 \end{aligned}\right\} \text{contradiction.}$$

KWELA SETS

THM (Kwela 2016) $\exists M \subseteq \mathbb{R}$ microscopic, $D \subseteq \mathbb{R}$, $|D| = \omega_1$ such that $M + D$ is not microscopic.

→ $\bigcup_{x \in D} M + x$ is not micro

→ add micro $(\mathbb{R}) = \omega_1$

KWELA SETS

THM (Kwela 2016) $\exists M \subseteq \mathbb{R}$ microscopic, $D \subseteq \mathbb{R}$, $|D| = \omega_1$, such that $M + D$ is not microscopic.

→ $\bigcup_{x \in D} M + x$ is not micro

→ $\operatorname{add} \operatorname{micro}(\mathbb{R}) = \omega_1$

QUESTION Is there a gauge ϕ such that $\operatorname{micro}(\mathbb{R}) = \mathcal{N}(\mathcal{H}^\phi)$?

→ Since $\operatorname{add} \mathcal{N}(\mathcal{H}^\phi) = \operatorname{add} \mathcal{N}$;
It is relatively consistent that the answer is NO.

→ Swaczyna 2023: Woodin Absoluteness Theorem yields
Modulo existence of many large cardinals, the answer is NO.

→ If $\mathcal{H}^\phi(E) = 0$, then $\exists P \subseteq \mathbb{R}$ perfect ... $\mathcal{H}^\phi(E \times P) = 0$
take for E a Kwela set: $E \times P$ is not microscopic

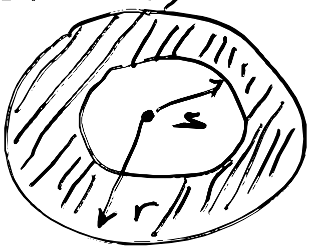
THM For every gauge ϕ, $\operatorname{micro}(\mathbb{R}) \neq \mathcal{N}(\mathcal{H}^\phi)$

KWELA SETS

- $I \subseteq \omega_+ \ldots$ $\underline{\mathrm{dens}}\, I = \underline{\lim} \dfrac{|I \cap n|}{n}$
- $E \subseteq X$ is subuicroscopic if $\forall \varepsilon > 0\ \exists I \subseteq \omega_+$
  - $\exists \langle \varepsilon^n \rangle$ - fin covr $\{E_n : n \in I\}$
  - $\underline{\mathrm{dens}}\, I = 0$
- $E \subseteq X$ is a KWELA SET if it is
  - microscopic
  - but not subuicroscopic

ONION SPACES

→ Let $x \in X$, $0<s<r$: $S(x,s,r) = B(x,r) \setminus B(x,s)$



"shell"

DEF (R. Vallin 1992) A metric space X is <u>strongly shell porous</u> at $x \in X$ if $\exists\ 0<s_n<r_n$

→ $\frac{s_n}{r_n} \to 0$

→ $\forall n \quad S(x,s_n,r_n) = \emptyset$

→ A ball $B(x,r)$ is an <u>onion</u> if $\exists \varepsilon>0 \ \forall s \le r \quad S(x,\varepsilon s,s) \neq \emptyset$.

DEF A space is an <u>onion space</u> if it is complete and strongly shell porous at no point.

(If each point of X is a center of an onion.)

ONION SPACES

THM Every onion space contains a family $\{K_\alpha : \alpha < \mathfrak{c}\}$ of Kunen sets that are pairwise separated.

$d(K_\alpha, K_\beta) > 0$

→ 

$\vdots$ 4^ω

$\mathrm{dist}(x,y) = \lambda^n$ where n is the first instance where $x_n \neq y_n$

$\mathbb{C}(4,\lambda)$

→ $\forall \lambda > 0$ $\mathbb{C}(4,\lambda)$ contains a Kunen set **HARD PART**

→ $\mathbb{C}(4,\lambda) \times \mathbb{C}(2,\lambda) \xleftrightarrow{\text{bi-Lipschitz}} \mathbb{C}(8,\lambda)$

→ Every onion space contains a bi-Lipschitz copy of $\mathbb{C}(8,\lambda)$

ONION SPACES

THM If $\{K_\alpha : \alpha < \omega_1\}$ are pairwise separated Kunen sets, then $\bigcup_{\alpha<\omega_1} K_\alpha$ is not microscopic.

Proof
- → $\forall \alpha\ \exists \varepsilon_\alpha > 0 \ldots$ each $\langle \varepsilon_\alpha^n \rangle$-fine cover of K_α has positive density.
  WLOG $\varepsilon_\alpha = \varepsilon$.
- → Suppose $K = \bigcup_\alpha K_\alpha$ is micro: $\exists \langle \varepsilon^n \rangle$-fine cover $\langle E_n : n \in \omega_+ \rangle$ of K.
- → Let $I_\alpha = \{ n \in \omega_+ : E_n \cap K_\alpha \neq \emptyset \}$: $\underline{\text{dens}}\, I_\alpha > 0$.
- → By separation: $\{ I_\alpha : \alpha < \omega_1 \}$ is AD family.
- → Every AD family with $\underline{\text{dens}}\, I_\alpha > 0$ is countable.

COR If X is an onion space, then
- → $\mathrm{add\ micro}(X) = \omega_1$
- → $\mathrm{cof\ micro}(X) = \mathfrak{c}$

ONION SPACES

- It is relatively consistent that $\exists X \subseteq \mathbb{R}$ $\operatorname{add} \operatorname{micro}(X) > \omega_1$
  ($\mathfrak{c}$-Sierpiński set)
- It is relatively consistent that $\exists X \subseteq \mathbb{R}$ $\operatorname{cof} \operatorname{micro}(X) < \mathfrak{c}$
  (Sierpiński set)

THM If X is a complete onion space, then
$\operatorname{micro}(X) \neq \mathcal{N}(\mathcal{H}^\phi)$ for every gauge ϕ.

ONION SPACES

→ It is relatively consistent that $\exists X \subseteq \mathbb{R}$ and $\mathrm{add}\ \mathrm{micro}(X) > \omega_1$ ($\mathfrak{c}$-Sierpinski set)

→ It is relatively consistent that $\exists X \subseteq \mathbb{R}$ $\mathrm{cof}\ \mathrm{micro}(X) < \mathfrak{c}$ (Sierpinski set)

THM If X is a complete onion space, then $\mathrm{micro}(X) \neq \mathcal{N}(\mathcal{H}^\phi)$ for every gauge ϕ.

→ A set $A \subseteq \mathbb{R}$ is null-additive if $\lambda(A+N)=0$ whenever $\lambda(N)=0$.

The existence of an uncountable null-additive set is relatively consistent (follows, e.g., from MA).

THM (MA) Let X be a complete onion space. For every family of gauges $\mathcal{G}$

$$\mathrm{micro}(X) \neq \bigcap_{\phi \in \mathcal{G}} \mathcal{N}(\mathcal{H}^\phi).$$

PRODUCTS AND SUMS

- M micro, $|A| > \omega$ $\Rightarrow$ $M \times A$ not micro
- M micro compact, A Sum $\Rightarrow$ $M \times A$ micro

PRODUCTS AND SUMS

- M micro, $|A| > \omega \implies M \times A$ not micro
- M micro compact, A Smz $\implies M \times A$ micro

THM $\mathcal{S} \subseteq \mathbb{S}$, $M \in \mathcal{E}(\mathcal{S})$, A Smz $\implies M \times A \in \mathcal{D}(\mathcal{S})$

$\longrightarrow$ but not $M \times A \in \mathcal{E}(\mathcal{S})$

(CH) $\exists$ A Smz $A + A = \mathbb{R}$. Let $M = \{x\}$. Then $A \times \{x\} \notin \mathcal{E}(\mathcal{S})$

otherwise $\exists K \supseteq A$, $K \in \mathcal{E}(\mathcal{S})$, hence $K + A \in \mathcal{D}(\mathcal{S})$.

But $K + A \supseteq A + A = \mathbb{R}$.

THM Let $\mathcal{S} \subseteq \mathbb{S}$ be closed, countable, downward directed. If $M \in \mathcal{E}(\mathcal{S})$ and Y a perfect set, then $\exists P \subseteq Y$ perfect such that

$\longrightarrow$ $M \times P \in \mathcal{E}(\mathcal{S})$

$\longrightarrow$ $M + P \in \mathcal{E}(\mathcal{S})$ (in a Polish group)

POROUS SETS

THM (Galvin–Mycielski–Solovay)

A set $X \subseteq \mathbb{R}$ is Smz $\iff$ $\forall M \subseteq \mathbb{R}$ meager $X+M$ is meager.

$E \subseteq X$ is porous if

$$\exists p>0 \ \forall x \in E \ \forall r>0 \ \exists y \in B(x,r) \quad B(y,pr) \cap E = \emptyset$$

POROUS SETS

THM (Galvin–Mycielski–Solovay)

A set $X \subseteq \mathbb{R}$ is Smz $\iff$ $\forall M \subseteq \mathbb{R}$ meager $X+M$ is meager.

→ $E \subseteq X$ is porous if

$$\exists p>0 \ \forall x \in E \ \forall r>0 \ \exists y \in B(x,r) \quad B(y,pr) \cap E = \emptyset$$

THM Let X be a Polish group.

If $M \subseteq X$ is micro and $P \subseteq X$ is porous, then $M+P \neq X$.

POROUS SETS

THM (Galvin–Mycielski–Solovay)

A set $X \subseteq \mathbb{R}$ is Smz $\iff$ $\forall M \subseteq \mathbb{R}$ meager $X+M$ is meager.

→ $E \subseteq X$ is porous if

$$\exists p > 0 \ \forall x \in E \ \forall r > 0 \ \exists y \in B(x,r) \ \ B(y, pr) \cap E = \emptyset$$

THM Let X be a Polish group.

If $M \subseteq X$ is micro and $P \subseteq X$ is porous, then $M+P \neq X$.

EX (Arturo Martínez) If $S \subseteq \mathbb{R}$ is Sierpiński set, then

$S+P \neq \mathbb{R}$ for each P porous and S is not micro

→ (Pawlikowski 1993) Sierpiński set is strongly meager

→ porous sets are Lebesgue null

→ Sierpiński set is not null $\Rightarrow$ not meager

CORO → micro $\not\Rightarrow$ porous

→ porous $\not\Rightarrow$ micro

CARDINAL INVARIANTS

- Uniformity $\quad \mathrm{non}\,\mathcal{J} = \min\{|A| : A \notin \mathcal{J}\}$
- Covering $\quad \mathrm{cov}\,\mathcal{J} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J},\ \bigcup\mathcal{A} = X\}$
- additivity $\quad \mathrm{add}\,\mathcal{J} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J},\ \bigcup\mathcal{A} \notin \mathcal{J}\}$
- cofinality $\quad \mathrm{cof}\,\mathcal{J} = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } \mathcal{J}\}$

$$\begin{tikzcd}
\mathrm{cov}\,\mathcal{N} \arrow[r] & \mathrm{non}\,\mathcal{M} \arrow[r] & \mathrm{cof}\,\mathcal{M} \arrow[rr] & & \mathrm{cof}\,\mathcal{N} \\
& \mathfrak{b} \arrow[u] & \mathfrak{d} \arrow[u] & & \\
\mathrm{add}\,\mathcal{N} \arrow[uu] \arrow[r] & \mathrm{add}\,\mathcal{M} \arrow[u] \arrow[r] & \mathrm{cov}\,\mathcal{M} \arrow[u] \arrow[r] & \mathrm{cov}^*\mathcal{M} \arrow[r] \arrow[d] & \mathrm{non}\,\mathcal{N} \arrow[uu] \\
& & & \min\{|A| : \exists B \in \mathcal{J}\ \ A+B = \mathbb{R}\} &
\end{tikzcd}$$

$\mathcal{M}$ meager sets

$\mathcal{N}$ null sets

CARDINAL INVARIANTS: DOMINATED SETS

- X separable: $\quad \mathrm{non}\,\mathcal{H}^\phi \geq \mathrm{cov}\,\mathcal{M}$
- X σ-totally bounded: $\quad \mathrm{non}\,\mathcal{H}^\phi \geq \mathrm{cov}^*\mathcal{M}$
- $X = \omega^\omega$: $\quad \mathrm{non}\,\mathcal{H}^\phi = \mathrm{cov}\,\mathcal{M}$
- X analytic, ϕ doubling: $\quad \mathrm{non}\,\mathcal{H}^\phi \leq \mathrm{non}\,\mathcal{N}$

THM If X is a not locally compact Polish group with an invariant metric, then for any $S \subseteq \mathbb{S}$ closed

$$\mathrm{non}\,\mathfrak{D}(S) = \mathrm{cov}\,\mathcal{M}.$$

THM Let X be separable and $S \subseteq \mathbb{S}$ closed. Then

$$\mathrm{cov}\,\mathcal{M} \leq \mathrm{non}\,\mathfrak{D}(S).$$

THM Let $X = 2^\omega$ or $\mathbb{R}$ and $S \subseteq \mathbb{S}$ be closed. If $S \cap \ell^1 \neq \emptyset$, then

$$\mathrm{cov}^*\mathcal{M} \leq \mathrm{non}\,\mathfrak{D}(S) \leq \mathrm{non}\,\mathcal{N}.$$

CARDINAL INVARIANTS: MICROSCOPIC SETS

THM (Głąb, Chrząszcz 2015) $\mathrm{non\ micro}(\mathbb{R}^n) = \mathrm{non\ micro}(2^\omega)$

THM Let X be analytic.

① If $\dim_H X > 0$, then $\mathrm{non\ micro}(X) \leq \mathrm{non\ micro}(2^\omega)$

② If X is doubling, then $\mathrm{non\ micro}(X) \geq \mathrm{non\ micro}(2^\omega)$

Proof ingredients:

→ A Hölder image of a microscopic set is micro.

① (Keleti–Máthé–Oz 2014) $\exists f : X \xrightarrow[\text{onto}]{\text{Hölder}} [0,1]$

② Assouad Embedding Thm: $X \xhookrightarrow{\text{Hölder}} \mathbb{R}^n$

CARDINAL INVARIANTS: MICROSCOPIC SETS

PROP There are invariant metrics ρ_1, ρ_2 on 2^ω such that

$$\mathrm{non}\ \mathrm{micro}(2^\omega, \rho_1) = \mathrm{cov}^* \mathcal{M}$$

$$\mathrm{non}\ \mathrm{micro}(2^\omega, \rho_2) = \mathrm{non}\ \mathcal{N}$$

THM It is relatively consistent that there is a compact Polish group X such that

$$\mathrm{non}\ \mathrm{micro}(X) \neq \mathrm{non}\ \mathrm{micro}(2^\omega)$$

QUESTIONS

Inscribing: Is the following statement true?
For each non-microscopic analytic (or G_δ) set $G \subseteq \mathbb{R}$ there is a compact non-microscopic set $K \subseteq G$.

Galvin-Mycielski-Solovay: We know that:
- $\to$ If $M \subseteq \mathbb{R}$ is micro, then $M+P \neq \mathbb{R}$ for each P porous
- $\to$ The converse fails.

Does the converse hold for M compact (G_δ, analytic)?

We know that:
- $\to$ For every Smz set $A \subseteq \mathbb{R}$, if M is σ-compact micro, then $A+M$ is micro.

Does this characterize Smz?